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K_σ-bounded sets and filters on ω

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K_σ - BOUNDED SETS AND FILTERS ON ω

by

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Abstract. We show how a property of "smallness" of subsets of ω^ω can be used to prove combinatorial result on filters of ω .

- A set $A \subset \omega^\omega$ is said K_σ-bounded if there exists a K_σ subset B of ω^ω (countable union of compact subsets of ω^ω) with $A \subset B$.

The family of K_σ-bounded subsets of ω^ω is a σ-ideal, and each non empty open set is not in it.

- Define, for F closed subset of ω^ω , F is superperfect if no non empty relative open set in F is relatively compact. By a derivation analogous to Cantor classical derivation, it can easily be proved that each closed F is the union of two disjoint sets F' and A, F' being superperfect (or $\emptyset$) and A K_σ-bounded (and this partition is unique).

It follows that each closed F satisfies the property (*)

(*) F is K_σ-bounded or contains a superperfect set

Theorem 1 (Kechris, S[T] Raymond)

Let $A \subset \omega^\omega$ be analytic. Then A satisfies (*).

This theorem can be extended to the following:

Theorem 2. 1) It is consistent with ZFC that all PCA sets satisfy (*)

2) It is consistent with ZF + DC (assuming the consistency of ZF + there is an inaccessible cardinal) that all subsets of ω^{ω} satisfy (*).

We use now these theorems in the case of filters on ω.

- A filter $\mathcal{F}$ on ω is free if it contains the Fréchet-filter $\mathcal{N}$. Then $\mathcal{F}$ is a subset of $[\omega]^{\omega}$, and when this space is equiped with the topology induced by 2^{ω}, we can speak of topological properties of filters. Moreover, as $[\omega]^{\omega}$ is homeomorphic to ω^{ω}, we can apply the previous results.

Theorem 3. Let $\mathcal{F}$ be a free filter on ω, and F be a closed (for the topology of $[\omega]^{\omega}$) subset of $\mathcal{F}$. Then F is a K_{σ}-set.

Corollary. 1) Every free analytic filter on ω is K_{σ}-bounded in $[\omega]^{\omega}$

2) Analogous statements of consistency follow from Theorem 2

Theorem 4. Suppose $\mathcal{F}$ is a K_{σ}-bounded free filter on ω. Then there exists a finite-to-one function h: $\omega \to \omega$ such that $h(\mathcal{F}) = \mathcal{N}$. (**)

Definition. $\mathcal{F}$ is said to be rare if for each partition of ω into finite sets, there is a selector of this partition which belongs to $\mathcal{F}$.

$\mathcal{F}$ is said to be rapid if for each increasing sequence

$(a_n)_{n\in\omega}$ of natural numbers, there is an $F \in \mathcal{F}$, such that if x_n is the n^{th} member of F, then $\forall n \quad a_n \leq x_n$.

Proposition. If a free filter on ω satisfies (**), it is not rapid, hence not rare.

Corollary. 1) No free analytic filter on ω is rare (Mathias)

2) Analogously, consistency statements follow from Theorem 2.

A paper on this subject will appear in the "Comptes-rendus du Congrès International de Logique de Clermont-Ferrand, Juillet 75".