Tomasz Byczkowski

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GAUSSIAN MEASURES ON L_p SPACES, $0 \leq p < \infty$

by

Tomasz BYCZKOWSKI

Let E be a real separable metric linear space, and let (Ω, σ, P) be a probability space. A mapping X from Ω to E is called a random element (r.e.) if it is measurable relative to the Borel σ-algèbra $\mathcal{B}$ in E and σ-algebra σ in Ω.

A random element X is called Gaussian if for every independent r.e.'s X_1, X_2 with the same distribution as X, the r.e.'s $X_1 + X_2$ and $X_1 - X_2$ are independent.

This definition has been used by Fréchet as one of two equivalent definitions of Gaussian random elements with values in a Banach space. This definition allows us to consider Gaussian random elements in metric linear spaces which admit no nontrivial continuous linear functional. The best known examples of such spaces are $L_p \equiv L_p(m)$ spaces, where m is the Lebesgue measure on $[0,1]$ and $0 \leq p < 1$. Of course, in such spaces the classical definition of Gaussian random elements by means of characteristic functional cannot be used. If the Borel σ-algebra in E is generated by all continuous linear functionals on E (for instance, if E is a separable Banach space) then this definition agrees with the usual one: a r.e.

X is Gaussian iff for every continuous linear functional f on E f(X) is a real Gaussian random variable.

Now, let $(T, \mathcal{F}, m)$ be a finite measure space. Let us assume that the Fréchet-Nikodym space is separable. Let $L_0 \equiv L_0(m)$ be the set of all equivalence classes of real functions that are defined on T and that are $\mathcal{F}$-measurable, with the norm

$$\|x\|_0 = \int_T \frac{|x(t)|}{1 + |x(t)|} \, m(dt) \quad .$$

and let $L_p \equiv L_p(m)$ be the subset of all $x \in L_0$ whose p-th power is m-integrable with the norm

$$\|x\|_p = \left(\int |x(t)|^p \, m(dt)\right)^{\varkappa}$$

where $\varkappa = \min(1, {}^1/p)$.

Now we shall need the notion of the measurable stochastic process.

Let $\{\xi(t);\ t \in T\}$ be a stochastic process; it is said to be measurable if the mapping ξ from $\Omega \times T$ into R defined by $(\omega, t) \longmapsto \xi(\omega, t)$ is measurable with respect to $\mathcal{G} \times \mathcal{F}$. Now, let us suppose that $\xi(\omega, \cdot) \in L_p$ P - a.e. Let $\tilde{\xi} : \Omega \longrightarrow L_p$ be the mapping defined as follows

$$\tilde{\xi}(\omega) = \begin{cases} \xi(\omega, \cdot) & \text{if } \xi(\omega, \cdot) \in L_p \\ 0 & \text{if } \xi(\omega, \cdot) \notin L_p \end{cases}$$

By the measurability of ξ and separability of L_p it follows that $\tilde{\xi}$ is a r.e. The probability distribution of $\tilde{\xi}$ will be denoted by μ_ξ.

We say that a strochastic process ξ is Gaussian if there is $T_0 \in \mathcal{F}$ such that $m(T_0) = 0$ and for every $t_1, t_2, \ldots, t_k \in T \setminus T_0$

$\langle \xi(t_1), \ldots, \xi(t_k) \rangle$ is a Gaussian random vector.

It seems to be interesting that there is a correspondence between Gaussian measures on L_p spaces and Gaussian measurable processes with sample paths in L_p. Namely, the following holds:

Theorem 1. Let μ be a probability measure on L_p space, $0 \leq p < \infty$. Then there is a measurable stochastic process $[\xi(t), t \in T]$ with the sample paths in L_p such that $\mu_\xi = \mu$. Moreover, if μ is Gaussian then ξ is Gaussian. On the other hand, if $[\xi(t), t \in T]$ is a Gaussian measurable stochastic process with the sample paths in L_p then μ_ξ is Gaussian.

In the probability theory it is always convenient to consider finite-dimensional rather than infinite-dimensional disstributions. There are some important cases in which this reduction can be done.

One of them is $C[0,1]$ space (of all real continuous functions defined on the unit interval with the supremum norm). It is well known that every probability measure on $C[0,1]$ is completely determined by the finite-dimensional distributions.

The second is $D[0,1]$ - the space of all real functions without discontinuities of the second kind.

There is one more example of such situation; this result seems to be new.

Theorem 2. Let μ_1 and μ_2 be two probability measures on L_p spaces. Let $\xi^{(1)}$, $\xi^{(2)}$ be two measurable stochastic processes inducing μ_1, μ_2, respectively. Then $\mu_1 = \mu_2$ iff there exists $T_0 \in \mathcal{F}$ such that $m(T_0) = 0$ and that if $t_1, \ldots, t_k \in T \setminus T_0$ then the random vectors

$$\langle \xi^{(1)}(t_1), \ldots, \xi^{(1)}(t_k) \rangle , \quad \langle \xi^{(2)}(t_1), \ldots, \xi^{(2)}(t_k) \rangle$$

have the same distributions.

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