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# A REMARK ON A PAPER BY KAREL PRIKRY

by

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Our note is a remark on a paper [2] by Prikry. In the present note, an ordinal is considered to be the set of the smaller ordinals. Cardinals are the initial ordinals. Moreover, we shall adopt the convention that $\xi$ and $\varphi$ denote ordinals, and that $\varkappa$ and $\lambda$ denote infinite cardinals. If $S$ is a set then $\mathcal{P}(S)$ denotes the family of all subsets of $S$. If $\mathcal{A}$ is a field of subsets of $S$, then by $\mathcal{J}(\mathcal{A})$ we denote the family of all $A \in \mathcal{A}$ such that for every $X \subset A$ we have $X \in \mathcal{A}$. A family of sets $\mathcal{R}$ is said to be $\varkappa$ - complete if for each family $\mathcal{F} = \{A_\xi : \xi < \lambda\} \subset \mathcal{R}$, where $\lambda < \varkappa$, $\bigcup \mathcal{F} \in \mathcal{R}$ as well. Let $\mathcal{J}$ be an ideal in $\mathcal{P}(S)$. A family $\mathcal{R} \subset \mathcal{P}(S)$ satisfies $(\varkappa.C.C.)(\mathcal{J})$ iff each family $\mathcal{F} \subset \mathcal{R}$ such that $F \cap G \in \mathcal{J}$ for all $F, G \in \mathcal{F}$, $F \neq G$ has power $< \varkappa$. We always assume $\emptyset \in \mathcal{J}$. If $\mathcal{J} = \{\emptyset\}$, then instead of $(\varkappa.C.C.)(\mathcal{J})$ we simply write $\varkappa$.C.C. The almost disjoint transversals hypothesis for $\omega_1$ is denoted by TH($\omega_1$). Let us recall that TH($\omega_1$) follows from Gödels axiom of constructability (for more information see [2]).

By modifications of Prikry's proofs and definitions in [2] we generalize theorems of [2] to the following form.

Theorem 1. Assume TH($\omega_1$). Let $\mathcal{M}$ be a family of $\omega_1$ - complete fields of subsets of $\omega_1$ such that $|\mathcal{M}| \leq \omega_1$ and for every $\mathcal{A} \in \mathcal{M}$ the family $\mathcal{A} - \mathcal{J}(\mathcal{A})$ satisfies $\omega_1$.C.C. and $\bigcup \mathcal{J}(\mathcal{A}) = \omega_1$.

Then for every $X \notin \bigcup\{\mathcal{J}(\mathcal{A}) : \mathcal{A} \in \mathcal{M}\}$ there exists a disjoint family $\{X_\xi : \xi \in \omega_1\} \subset \mathcal{P}(X)$ such that $X_\xi \notin \mathcal{A}$ for every $\mathcal{A} \in \mathcal{M}$ and every $\xi \in \omega_1$.

Theorem 2. Let $\varkappa$ and $\lambda$ be regular cardinals such that $\lambda \geqslant \varkappa \geqslant \omega_1$ and let $\mathcal{M}$ be a countable family of $\varkappa$-complete fields of subsets of $S$ such that for every $\mathcal{A} \in \mathcal{M}$ the family $\mathcal{A} - \mathcal{J}(\mathcal{A})$ satisfies $\varkappa$.C.C. . Suppose $\mathcal{J} \subset \bigcap\{\mathcal{J}(\mathcal{A}) : \mathcal{A} \in \mathcal{M}\}$ is a $\varkappa$- complete ideal in $\mathcal{P}(S)$ such that for every $\mathcal{A} \in \mathcal{M}$ and every $X \in \mathcal{P}(S) - \mathcal{J}(\mathcal{A})$ the family $\mathcal{P}(X) - X \cap \mathcal{J}(\mathcal{A})$ does not satisfy $(\lambda$.C.C.$)(\mathcal{J})$. Then for every $X \subset S$ there exists a family $\{X_\xi : \xi \in \lambda\} \subset \mathcal{P}(X)$ such that

(i) $X_\xi \cap X_\varphi \in \mathcal{J}$ for every $\xi \neq \varphi$ , $\xi, \varphi \in \lambda$ , and

(ii) for every $\xi \in \lambda$ , every $\mathcal{A} \in \mathcal{M}$ and every $A \in \mathcal{A}$ we have, if $X_\xi - A \in \mathcal{J}(\mathcal{A})$ , then $X - A \in \mathcal{J}(\mathcal{A})$.

Theorem 3. Let $\lambda$ be a regular cardinal and let $\mathcal{M}$ be a finite family of fields of subsets of $S$ such that for every $\mathcal{A} \in \mathcal{M}$ the family $\mathcal{A} - \mathcal{J}(\mathcal{A})$ satisfies $\omega_0$.C.C. . Suppose $\mathcal{J} \subset \bigcap\{\mathcal{J}(\mathcal{A}) : \mathcal{A} \in \mathcal{M}\}$ is an ideal in $\mathcal{P}(S)$ such that for every $\mathcal{A} \in \mathcal{M}$ and every $X \in \mathcal{P}(S) - \mathcal{J}(\mathcal{A})$ the family $\mathcal{P}(X) - X \cap \mathcal{J}(\mathcal{A})$ does not satisfy $(\lambda$.C.C.$)(\mathcal{J})$. Then for every $X \subset S$ there exists a family $\{X_\xi : \xi \in \lambda\} \subset \mathcal{P}(X)$ satisfying the conditions (i) and (ii) of Theorem 2.

In [2] Prikry has proved theorems 1 and 2 in the case if $\mathcal{A}$ consists only $\omega_1$ - complete fields on which it is possible to define complete probabilities which vanishe on all finite sets. Theorem 1 of this note generalizes Theorem 1 of [2], and Theorem 2 of this note generalizes Theorem 2 of [2]. Our generalization of Theorem 2 of [2] may be motivated by the fact that it gives a common generalization and strenghtening of known theorems for measures (see Sierpiński [4] and Prikry [2] ), for outer measures (see Popruženko [1] ), and for the category of Baire (see Sierpiński [3] ).

An extended form of this note will be submitted for publication elsewhere.

## References

[1] J. Popruženko, Sur une propriété d'une classe de measures abstraites, Colloquium Mathematicum 4(1957), p. 189-194.

[2] Karel Prikry, Kurepa's hypothesis and a problem of Ulam on families of measures, Monatshefte für Mathematik 81, (1976), p. 41-57.

[3] W. Sierpiński, Sur les ensembles partout de deuxième catégorie, Fundamenta Mathematicae 22(1934), p. 1-3.

[4] -, Sur une propriété des ensembles linéaires quelconques, Fundamenta Mathematicae 23(1934), p. 125-134.