P. Kenderov

A general topology approach to the study of differentiability of convex functions in Banach spaces

In: Zdeněk Frolík (ed.): Abstracta. 6th Winter School on Abstract Analysis. Czechoslovak Academy of Sciences, Praha, 1978. pp. 55–59.

Persistent URL: http://dml.cz/dmlcz/701121

A GENERAL TOPOLOGY APPROACH TO THE STUDY OF DIFFERENTIABILITY OF CONVEX FUNCTIONS IN BANACH SPACES [*)]

by P.Kenderov (Sofia)

(An abstract from a report presented at the 6-th Winter School on Abstrct Analysis, Czechoslovakia 1978)

Let E be a Banach space and $f : E \longrightarrow R$ be a real valued convex and continuous function whose domain $D(f) = \{x \in E : f(x) < +\infty\}$ has interior points: $\mathrm{int}D(f) \neq \emptyset$. f is said to be Gâteaux differentiable at some point x_o in $\mathrm{int}D(f)$ if there exists x^* in E^* (:= the space of all linear continuous functionals in E) such that

$$\lim_{t \to 0} (f(x_o + th) - f(x_o))/t = x^*(h) \quad \text{for every } h \text{ in } E.$$

f is said to be Fréchet differentiable at x_o if the above limit is uniform with respect to all h, $\|h\| \leqq 1$. It is known that both Gâteaux and Fréchet differentiability of f at x_o can be easily expressed by means of the so called subgradient ∂ of f (the latter is a multivalued mapping $\partial : E \longrightarrow E^*$ assigning to each x_o from E the set $\partial(x_o) := \{x^* \in E : \langle x - x_o, x^* \rangle \leqq f(x) - f(x_o)$ for every x in $E\}$).

PROPOSITION. i) f is Gâteaux differentiable at x_o iff the set $\partial(x_o)$ is a singleton (i.e. consists of only one element).

ii) f is Fréchet differentiable at x_o iff $\partial(x_o)$ is a singleton and for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|x - x_o\| < \delta$ implies $\partial(x) \subset B(\partial(x_o), \varepsilon)$. Here $B(\,.\,, \varepsilon)$ denotes the ball with center at "." and of radius ε.

---

*) A part of this work was done while the author was at the University of Frankfurt am Main as a research fellow of the Alexander von Humboldt Foundation.

Therefore, if we want to prove that a given convex function $f : E \longrightarrow R$ is Gâteaux differentiable at the points of some dense G_δ subset of E, it is enough to prove somehow that the multivalued mapping $\partial : E \longrightarrow E^*$ is single-valued at the points of such a subset of E. In order to have Fréchet differentiability on dense G_δ subset of E, it is enough to prove, in addition to the Gâteaux case, that $\partial : E \longrightarrow E^*$ satisfies the condition in ii) for every point from some dense G_δ subset of E.

Now we would like to outline how a general topology phenomenon (concerning multivalued mappings) could be applied to the present situation so as to prove some known results about differentiability "almost everywhere" of convex functions defined in Banach spaces. Recall first that the multivalued mapping $F : X \longrightarrow Y$ from the topological space X into the topological space Y is said to be upper semicontinuous (lower semicontinuous) at $x_o \in X$ if for every open $U \subset Y$, $U \supset F(x_o)$ $(U \cap F(x_o) \neq \emptyset)$ there exsits an open V in X, $V \ni x_o$, such that from $x \in V$ it follows that $F(x) \subset U$ $(F(x) \cap U \neq \emptyset)$ *). One aspect of the general topology phenómenon we have in mind, is expressed by the following theorem of Kuratowski ([4] ,p.79) - Fort [2] :

If $F : X \longrightarrow Y$ is an usc (lsc) multivalued mapping with compact images, acting from the topological space X into the metrizable space Y, then F has to be lsc (usc) at the points of some dense G_δ subset of X.

THEOREM (Moreau [6]). If $f : E \longrightarrow R$ is a continuous con-

*) We will write for brevity "usc" (lsc) instead of "upper semicontinuous" (lower semicontinuous").

vex function, then $\partial : E \longrightarrow E^*$ is usc with respect to the norm topology in E and weak* topology in E^* (i.e. $\partial : (E, \|.\|) \rightarrow \longrightarrow (E^*, w^*)$ is usc) at every point of $\operatorname{int} D(f)$. Moreover, for every x_o in $\operatorname{int} D(f)$ there exists an open $V \ni x_o$ such that $\partial(V) = \bigcup_{(x \in V)} \partial(x)$ is a bounded subset of E^*.

This theorem combined with the above mentioned theorem of Kuratowski-Fort implies the following corollary:

COROLLARY. If E is a separable Banach space and $f : E \longrightarrow R$ is continuous convex function, then $\partial : (E, \|.\|) \longrightarrow (E^*, w^*)$ is lsc at the points of some dense G_δ subset of $\operatorname{int} D(f)$.

The last (but decisive) step now is contained in the next

PROPOSITION (Kenderov [3], Robert [7]). If $\partial : (E, \|.\|) \longrightarrow \longrightarrow (E^*, w^*)$ is lsc at the point $x_o \in X$, then $\partial(x_o)$ is a singleton (and therefore $f : E \longrightarrow R$ is Gâteaux differentiable at x_o).

As a corollary we get the classical result of Mazur [5] that every continuous convex function defined in the separable Banach space E must be Gâteaux differentiable at the points of some dense G_δ subset of E.

Since all the argument above depends in fact only on the monotonicity properties of $\partial : E \longrightarrow E^*$, we can apply (in exactly the same way) the same general topology approach in order to prove the result of Zarantonello [10] that every multivalued monotone operator $T : E \longrightarrow E^*$ acting in the separable Banach space E must be single-valued at the points of some dense G_δ subset of $\operatorname{int} D(T)$, where $D(T) = \{ x \in E : T(x) \neq \emptyset \}$. Taking as a start point another expression of the same general topology fenomenon and applying it (in a similar way) to more generl situations, we can get some new results as well as new proofs of some known results concerning differentiability "almost everywhere" of convex functions (Asplund[1]), concerning single-valuedness "almost everywhere" of multivalued monotone operators and concerning uniqueness "almost everywhere" of the best approximations in Banach spaces (Stechkin[9]).

R E F E R E N C E S

[1] E.Asplund, Fréchet differentiability of convex functions, Acta Math. 121(1968), 31-47

[2] M.K.Fort, Points of continuity of semi-continuous functions, Public.Mathem.,Debrecen,2 (1951),p.100-102

[3] P.Kenderov, Continuity of multivalued mappings and differentiability of convex functions, Ann l'Univ.Sofia 66(1971-72) p.163-168 .

[4] K.Kuratowski,Topology vol.2, Moskow 1969

[5] S.Mazur, Über konvexe Mengen in linearen normierten Räumen, Studia Math.,4(1933), 70-84 .

[6] J.J. Moreau, Semi-continuité du sous-gradient d'une fonctionelle, C.R.Sc.Paris, 260(1965), 1067-1070

[7] R.Robert,Points de continuité de multi-application semicontinues supérieurement, C.R.Acad.Sc.Paris,278(1974) A 413-415.

[8] R.Robert, Une généralization aux operateurs monotone des théorêmes de différentiabilité d'Asplund, C.R.Acad.Sc.Paris, 278 (1974), A 1189 - 1191.

[9] S.B.Stechkin,Approximative properties of Banach space subsets, Rev.Roum.Math.Pures et Appl.,8 (1963), 5-8 .

[10] E.Zarantonello, Dense single-valuedness of monotone operators, Israel J.Math. 15(1973),158-166

Permanent address of the author:

Institute of Mathematics
Bulgarian Academy of Sciences
1000 Sofia, P.O.Box 373
P.R.BULGARIA

Temporary address (till November 1978):

Fachbereich Mathematik
Jochann Wolfgang Goethe- Universität
Robert-Mayer-Straße 6-10
6000 Frankfurt am Main
F.R. GERMANY