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MODEL THEORETIC APPROACH TO TOPOLOGICAL FUNCTORS

by

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An infinitary first-order language $L_{\infty,\infty}$ has a class of function symbols, a class of relation symbols and a class of variables. Infinitary function and relation symbols are admitted and infinitary conjunctions and quantifiers, as well. An infinitary Horn theory H is a theory of $L_{\infty,\infty}$ whose axioms are all of the form (where we will assume that the following formulas all have their free variables universally quantified in front):

(1) φ where φ is an atomic formula

(2) $\bigwedge_{i \in I} \varphi_i \longrightarrow \theta$ where φ_i, $i \in I$ and θ are atomic formulas.

In addition, we assume that H satisfies some smallness conditions (details can be found in [2]). In [2] there is given a characterization of forgetful functors from a category of models of an infinitary Horn theory into sets. This characterization can be restated in terms of semi-topological functors in the sense of Hoffmann-Tholen (see [3]) as follows.

<u>Theorem 1</u>: Let $U: \mathcal{A} \longrightarrow \mathrm{Set}$ be a functor. Then the following conditions are equivalent:

(3) U is equivalent to a forgetful functor from a category of models of an infinitary Horn theory H

(4) (a) $\mathcal{A}$ is co-well-powered

(b) U is semi-topological and U-quotients of epi-sinks are epi.

The condition (4)(b) means that for any sink $(f_i: UA_i \longrightarrow X)_{i \in I}$ there exists a sink $(g_i: A_i \longrightarrow A)_I$ and a map $e: X \longrightarrow UA$ with $Ug_i = e.f_i$ for each $i \in I$ and such that for any sink $(h_i: A_i \longrightarrow B)_I$ and any map $t: X \longrightarrow UB$ with $Uh_i = t.f_i$ for each $i \in I$ there exists a unique morphism $k: A \longrightarrow B'$ with $Uk.e = t$ and $t.g_i = h_i$ for each $i \in I$. In addition to it, if $(f_i)_I$ is epi, then e have to be epi.

As a consequence, we can get the model theoretic interpretation of Herrlich's (E,M)-topological set functors (see [1]).

Theorem 2: Let $U: \mathcal{A} \longrightarrow$ Set be a functor. Then the following conditions are equivalent:

(5) U satisfies (3) and (a) contains relation symbols only (i.e. no function and constant symbols)

(b) H can be axiomatized in such a way that in any axiom (1) φ is equal to r(....) where r is a relation symbol (and not the equality of two variables)

(6) $\mathcal{A}$ is co-well-powered and U is (epi,mono-source)-topological.

Note, that (5)(b) only ensures that H has a model with more than one element. It arises a problem of a syntactical description of absolutely topological functors in the sense of [1].

Conjecture: A functor U from a co-well-powered category $\mathcal{A}$ into sets is absolutely topological iff U satisfies (5) and H can be axiomatized in such a way that in any axiom (2) θ is equal to r(....) where r is a relation symbol (and not to the equality of two variables).

It is easy to verify that any such functor U is absolutely topological. As an example, the antisymmetry $(x \le y) \wedge (y \le x) \rightarrow (x = y)$ causes that the forgetful functor from ordered sets is not absolutely topological.

References:

[1] H. Herrlich, Topological functors, Gen. Topol. Appl. 4 (1974), 125-142.

[2] J. Rosický, Categories of models of infinitary Horn theories, to appear in Arch. Math. (Brno), 4 (1978).

[3] W. Tholen, Semi-topological functors I., Seminarberichte, Fernuniversität Hagen 3 (1977), 13-48.