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In: Zdeněk Frolík (ed.): Abstracta. 7th Winter School on Abstract Analysis. Czechoslovak Academy of Sciences, Praha, 1979. pp. 48--52.

Persistent URL: http://dml.cz/dmlcz/701147

COMMUTATIVE HARMONIC ANALYSIS AND BANACH SPACES

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Preliminaries. Let G be a compact abelian group, Γ its dual, m-the normalized Haar measure on G. The symbols $C(G)$, $L^p(G)$ $(0<p<\infty)$ denote as usual the spaces of the continuous scalar valued functions on G, respectively of m-equivalence classes of measurable p-absolutely integrable functions on G. For $a\in G$, τ_a denotes the operator of translation by a acting on functions on G by the rule $\tau_a f(x) = f(x-a)$.

A linear space X of (equivalence classes of) functions is called translation invariant if $\tau_a(X)\subset X$ for all $a\in G$. A linear operator acting between translation invariant spaces is translation invariant if it commutes with all τ_a

A translation invariant Banach space X is regular if

(a) X consists of equivalence classes of absolutely integrable functions on G; the inclusion $X \hookrightarrow L^1(G)$ is a one to one continuous operator;

(b) $\tau_a : X \to X$ is an isometry for all $a\in G$.

(c) given $f\in X$ the map $a \to \tau_a f$ is a continuous function from G into X.

The elements of Γ are called characters. A trigonometric polynomial is a finite linear combination of the characters. "Measure" means here a complex valued Borel measure on G whose total variation is bounded. For $f\in L^1(G)$, resp. for a measure μ the Fourier transforms are the functions $\hat{f}$, resp. $\hat{\mu}$ on Γ defined by $\hat{f}(\gamma) = \int f\bar{\gamma}\,dm$, resp. $\hat{\mu}(\gamma) = \int \bar{\gamma}\,d\mu$ for $\gamma\in\Gamma$.

For a $\Lambda\subset\Gamma$, C_Λ denotes the closed linear subspace of $C(G)$ generated by Λ.

Main results presented in the lectures

Lecture I

Theorem 1.1. Let $\Lambda \subset \Gamma$. Then

1^o Λ is a Cohen set (i.e. there is a measure whose Fourier transform is the characteristic function of Λ) iff C_Λ is an $\mathcal{L}_\infty$ space in the sense of Lindenstrauss and Pelczyński [LP].

2^o Λ is a Sidon set (i.e. there exists a $k > 0$ such that for every trigonometric polynomial $f = \sum_{\gamma \in \Lambda} c_\gamma \gamma$, $\|f\|_\infty \geq \sum_\gamma |c_\gamma| k$) iff C_Λ is an $\mathcal{L}_1$ space in the sense of [LP].

Part 2^o is due to Varopoulos [V]. The proof presented in the Lecture bases on the following (cf. [KP]).

Proposition 1.2. Let C_Λ be such that every finite dimensional operator from the dual space of C_Λ into C_Λ factors through a Hilbert space. Then Λ is a Sidon set.

Corollary 1.3. (cf. [KP] and [Pi]). $\Lambda \subset \Gamma$ is a Sidon set iff C_Λ is a Banach space of cotype 2 (cf.e.g. [M] for the definition of the cotype).

Lecture II

Theorem 2.1. Every regular translation invariant Banach pace X has the invariant uniform approximation property; precisely for every $\epsilon > 0$ there is a function $m \to q_\epsilon(m)$ such that given a finite dimensional translation invariant subspace E o X there exists a translation invariant operator u_E such that

(1) $u_E(e) = e$ for $e \in E$, (2) $\|u_E\| < 1 + \epsilon$,

(3) $\dim u_E(X) \leq q_\epsilon(\dim E)$.

Theorem 2.1 follows immediately (in fact is equivalent to) from the next one

Theorem 2.2. For every $\epsilon > 0$ there is a function $m \to q_{\epsilon}(m)$ such that given a finite set $M \subset \Gamma$ there is a trigonometric polynomial g_E such that

(i) $\hat{g}(\gamma) = 1$ for $\gamma \in M$,

(ii) $\|g_E\|_1 < 1 + \epsilon$,

(iii) $|S(g)| \le q_{\epsilon}(|M|)$.

Here $S(g) = \{\gamma \in \Gamma : \hat{g}(\gamma) \neq 0\}$ and $|A|$ denotes the number of elcments of a finite set A.

Theorem 2.1 and 2.2 are taken from the paper by M. Bożejko and A. Pelczyński [BP].

Lecture III

Definition 3.1. A set $\Lambda \subset \Gamma$ is a Marcinkiewicz set if the orthogonal projection $P_\Lambda : L^2(G) \to L^2(G)$, defined by $P_\Lambda f = \sum_{\gamma \in \Lambda} \hat{f}(\gamma)\gamma$, regarded as the operator on trigonometric polynomials is $(1,p)$ bounded for some (equivalently for all) p with $0 < p < 1$, i.e. there is a $k > 0$ such that

$$\left(\int_G |P_\Lambda (f)|^p dm\right)^{\frac{1}{p}} \le k \int_G |f| dm \quad \text{(f-trigonometric polynomial)}$$

Recall that an operator $u : X \to Y$ (X,Y-Banach spaces) is said to be p-absolutely summing $(0 < p < \infty)$ if there exists a constant $C > 0$ such that for every finite set $F \subset X$

$$\sum_{x \in F} \|ux\|^p \le C \sup \sum_{x \in F} |x^*(x)|^p$$

where the supremum is taken over all x^* in the unit ball of the dual of X.

Theorem 3.2. [KP]. If Λ is a Marcinkiewicz set then every translation invariant operator $u : L^2(G) \to C_\Lambda$ has the one--absolutely summing adjoint.

Theorem 3.2 can be regarded as a generalization for translation invariant operators of Grothendieck's "Fundamental Theorem in Metric Theory of Tensor Products" (cf. [G],[LP]). The proof presented in the lecture bases upon the following fact essentially proved in [KP] .

<u>Theorem 3.3.</u> Let Λ be a Marcinkiewicz set, $0<p<1$, X-a regular translation invariant Banach space. Then every p-absolutely summing translation invariant operator $u : C_\Lambda \longrightarrow X$ is integral; precisely there exists a bounded linear operator $v : L^1(G) \longrightarrow X$ such that the diagram

$$\begin{tikzcd} C(G) \arrow[r, "i"] & L^1(G) \arrow[d, "v"] \\ C \arrow[u, "j"] \arrow[r, "u"] & X \end{tikzcd}$$

is commutative, where j is the natural isometric embedding and i is the natural injection which assigns to each f in $C(G)$ its m-equivalence class in $L^1(G)$.

References

[BP] M. Bożejko and A. Pelczyński: An analogue in commutative Harmonic Analysis of the uniform bounded approximation property of Banach spaces, Seminaire d'Analyse Fonctionelle 1978-9, Exposé No. IX, Ecole Polytechnique, Palaiseau 1979

[G] A. Grothendieck: Résumé de la théorie métrique des produits tensoriels topologiques, Bol.Soc.Matem., 8 (1956), 1-79

[KP] S. Kwapień and A. Pelczyński: Absolutely summing operators and translation invariant spaces of functions on compact abelian groups, Math.Nachr., to appear; Inst. of Math.Polish.Acad.Sci. Preprint, Warsaw 1978

[LP] J. Lindenstrauss and A. Pelczyński: Absolutely summing operators in $\mathcal{L}_p$ spaces and their applications, Studia Math., 29 (1968), 275-326

[M] B. Maurey: Théorèmes de factorisations pour les opérateurs linéaires à valeurs dans les espaces L^p , Asterisque, 11 (1974)

[Pi] G. Pisier: Seminaire d'Analyse Fonctionelle 1977-8, Ecole Polytechnique, Palaiseau 1978

[V] N.T. Varopoulos: Sous espaces de $C(G)$ invariant par translation et de type α_1, Seminaire Maurey--Schwartz 1975/6, Expose 12, Ecole Polytechnique, Palaiseau 1976

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