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SIFTING INFINITE - DIMENSIONAL COMPACTA BY THE LUSIN SIEVE

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1. By a compactum we shall understand a compact metrizable space. Recall that a compactum X is countably dimensional if X is the union of countably many zero dimensional subsets and X is weakly infinite dimensional, if for each infinite sequence $(A_1,B_1), (A_2,B_2),\ldots$ of pairs of disjoint closed sets in X there are partitions L_i in X between A_i and B_i such that $\bigcap_i L_i = \emptyset$ (see [A-P] or [N]).

The class of countably dimensional compacta is contained in the class of weakly infinite dimensional compacta, while the question whether the inverse inclusion holds is the well known problem of Aleksandrov.

Compacta which are not weakly infinite dimensional we shall call strongly infinite dimensional.

Countably dimensional compacta can be classified inductively as follows: let $\operatorname{Ind} X \leq \alpha$ if for each pair of disjoint closed sets in X there is a partition L separating these sets such that $\operatorname{Ind} L < \alpha$ and let $\operatorname{Ind} X$ be the least α with $\operatorname{Ind} X \leq \alpha$; such an α exists if and only if X is countably dimensional and then $\alpha < \omega_1$.

2. Let X be a compactum. Call a sequence $\mathcal{S} = \{(A_1,B_1), (A_2,B_2),\ldots\}$ of pairs of disjoint closed sets in X a basic

sequence if for each pair of disjoint closed sets (A,B) in X the inclusions $A \subset A_i$ and $B \subset B_i$ hold simultaneously for infinitely many indices i.

Denote by $\mathrm{Fin}\,\omega$ the set of all finite subsets of natural numbers ω and let $\prec$ be the ordering of $\mathrm{Fin}\,\omega$ inverse to the lexicographic ordering (i.e. $\sigma \prec \tau$ if for some $n \in \omega$ we have $\sigma(i) = \tau(i)$ if $i < n$ and $\sigma(n) = 1$, $\tau(n) = 0$). Put

$$M_X^{\varphi} = \Big\{ \sigma \in \mathrm{Fin}\,\omega : \text{if } L_i \text{ is a partition in } X \text{ between } A_i \text{ and } B_i, \text{ then } \bigcap_{i \in \sigma} L_i \neq \emptyset \Big\}.$$

We have then

(a) M_X^{φ} is well ordered by $\prec$ if and only if X is weakly infinite dimensional;

(b) if X is weakly infinite dimensional, then the order type of M_X^{φ} does not depend on the choice of the basic sequence φ.

Having (b) in mind we define for a weakly infinite dimensional compactum X : index X = order type M_X^{φ}, where φ is a basic sequence in X.

Note that if $Y \subset X$, then index $Y \leq$ index X.

3. Denote by $\underline{H}$ the hyperspace of the Hilbert cube I^{ω}, i.e. $\underline{H}$ is the space of all compacta in I^{ω} endowed with the Hausdorff metric.

There exists a Lusin sieve $\underline{W}$ consisting of closed subsets of $\underline{H}$ (see [K]), defined in a natural way by means of a basic sequence in I^{ω}, such that

(a) the set $L(\underline{W})$ sifted by the sieve $\underline{W}$ is exactly the set of all strongly infinite dimensional compacta in I^{ω};

($X \in \underline{H}$ $L(\underline{W})$ then the Lusin index of X with resp ct t _ (e $[K]$) coincid s vith the topological in r i de X

I an e arily veri‑i th t for very $\alpha < \omega_1$ t $\{ \in \underline{H}$ Ind $X \leq \alpha \}$ is analytic and since the Lusin n is bounded on analytic sets we have

$$\sup \{ \text{index } X : \text{Ind } X \leq \alpha \} < \omega_1 \quad \text{for} \quad \alpha < \omega_1 .$$

Question 1. Is it true that Ind is bounded on each set of countably dimensional compacta with bounded index?

If not, Aleksandrov's problem has the negative solution, as we have

Theorem 1. For a family $\underline{F}$ of weakly infinite dimensional compacta the following conditions are equivalent:

(i) there exists a weakly infinite dimensional compactum containing topologically each compactum from $\underline{F}$;

(ii) $\sup \{ \text{index } X : X \in \underline{F} \} < \omega_1$.

Another consequence of the boundedness of the Lusin index on analytic sets is

Theorem 2. Let $\underline{F}$ be an upper semi-continuous decomposition of an arbitrary compactum X into weakly infinite dimensional compacta. Then $\sup \{ \text{index } X : X \in \underline{F} \} < \omega_1$.

Question 2. Assume that $\underline{F}$ above consists of countably dimensional compacta. Is it true that $\sup \{ \text{Ind } X : X \in \underline{F} \} < \omega_1$?

The negative answer would provide the negative answer to Question 1 and thus, as we observed, the negative solu ion o Aleksandrov's problem.

4. In thi section ve shall a ply the noti n o

a concrete situation, answering a question of D. Henderson raised in [H].

Henderson defined, by the transfinite induction, "cubes" H_α and their "boundaries" ∂H_α of order $\alpha < \omega_1$. If $i < \omega$, then H_i is the i-dimensional cube and ∂H_i is its boundary. Assume that for $\xi < \alpha$ we have defined H_ξ, ∂H_ξ and points $p_\xi \in \partial H_\xi$. If $\alpha = \xi + 1$ put $H_{\xi+1} = H_\xi \times I$, $\partial H_{\xi+1} = (\partial H_\xi \times I) \cup (H_\xi \times \{0,1\})$ (where I is the unit interval) and $p_{\xi+1} = (p_\xi, 0)$; if α is limit, let K_ξ be the union of H_ξ and a half open arc whose origin p_ξ is its only common point with H_ξ, let H_α be the one-point compactification of the free union of all K_ξ for $\xi < \alpha$, $\partial H_\alpha =$ $= H_\alpha \setminus \bigcup_{\xi<\alpha} (H_\xi \setminus \partial H_\xi)$ and let p_α be the compactifying point.

Henderson showed that H_α are AR-compacta and defined essential mappings into H_α extending the classical notion as follows: a continuous $f : X \to H_\alpha$ is essential provided that if $g : X \to H_\alpha$ is a continuous map which coincides with f on the set $f^{-1}(\partial H_\alpha)$, then $g(X) = H_\alpha$. Henderson proved that if a countably dimensional compactum X admits an essential map onto H_α then $\operatorname{Ind} X \geq \alpha$ and asked, whether a compactum which admits an essential map onto each compactum H_α is strongly infinite dimensional ?

One can verify by the transfinite induction that if a weakly infinite dimensional compactum X admits an essential map onto H_α then index $X \geq \alpha$ and this yields immediately (see sec. 1 (a)) the affirmative answer to the question of Henderson.

5. Finally, let us mention a result which, although not related to the notion of index, has a proof based on the classical theory of analytic sets.

Yu. Smirnov [S] defined by transfinite induction compacta $S_1, S_2, \ldots, S_\xi, \ldots, \xi < \omega_1$ with $\operatorname{Ind} S_\xi = \xi$ as follows: S_1 is the unit interval I, $S_{\xi+1} = S_\xi \times I$ and if ξ is a limit ordinal, then S_ξ is the one-point compactification of the free union of all S_η for $\eta < \xi$.

Theorem. If a complete separable metrizable space X contains topologically all Smirnov's compacta, then X contains topologically the Hilbert cube.

In particular, we have

Corollary. If X is a complete separable metrizable space such that there is a continuous injection of the cone over X into X, then X contains topologically the Hilbert cube.

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