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On the geometrical structure of Lebesgue nullsets

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## SEVENTH WINTER SCHOOL (1979)

On the geometrical structure of Lebesque nullsets

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A complete description in geometrical terms of the class of Lebesgue nullsets is impossible. Of course, we have the definition in terms of coverings with balls but what we are after are local or pointwise characterizations. We start with a simple result:

Theorem 1. Let $Z$ be a subset of $\mathbb{R}^2$ , say. Assume that, for every $x \in Z$ , there exists a conesection with toppoint in $x$ and disjoint from $Z$ . Then $|Z| = 0$ .

When we require that a conesection be disjoint with $Z$ , we really mean that the interior be disjoint with $Z$ . $|\cdot|$ denotes Lebesgue measure.

Theorem 1 follows immediately from Lebesgues density theorem. However, as the density theorem is deduced from the Vitali theorem, we consider the Vitali theorem as the basic fact. Therefore, we will show how Theorem 1 may be obtained from Vitali's theorem (in the form below, due to Banach, I think).

Let $A \subseteq \mathbb{R}^2$ and let $\mathcal{S}$ be a system of closed balls in $\mathbb{R}^2$ . We define the homothetical Vitali system $\mathcal{V}_{\text{hom}}$ by the requirement

$$(A,\mathcal{S}) \in \mathcal{V}_{hom} \iff \forall x \in A\ \exists \infty \geq c > 0\ \forall \delta\ \exists B = B[y,r] \in \mathcal{S}:$$

$$B \subseteq B[x,\delta]\ ,\ r \geq c \cdot d(y,x).$$

<u>Theorem</u> 2. <u>(The classical Vitali theorem)</u>.
$(A,\mathcal{S}) \in \mathcal{V}_{hom} \Rightarrow \exists\ (B_n)$ <u>disjoint from</u> $\mathcal{S}$ <u>such that</u>
$|A \setminus \cup B_n| = 0$.

Clearly, Theorem 2 implies Theorem 1 (put $\mathcal{S} = \{B$ closed ball$|$ $\overset{o}{B} \cap Z = \emptyset\}$).

We shall now investigate what happens if the convex conesections are replaced by non-convex "conesections". Let $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ be monotone and assume that $\eta(\delta) \leq \delta\ \forall \delta > 0$. Rotating the $\eta$-curve around some half line in $\mathbb{R}^N$ emerging from a point $x \in \mathbb{R}^N$, and intersecting with a ball with center at $x$, we obtain what we shall call an $\eta$-<u>section</u> <u>at</u> $x$. Thus, if $\eta$ is of the form $\eta(\delta) = c \cdot \delta$, we get a cone.

We define a class (*) of $\eta$-functions by the requirement

$$(*) \qquad \int_0^1 \left(\frac{\eta(\delta)}{\delta}\right)^N \frac{d\delta}{\delta} = \infty .$$

Here, $N$ is the dimension of the euclidean space we are working in. We remark that

$$\eta \in (*) \Leftrightarrow \sum_0^\infty \left(\frac{\eta(2^{-n})}{2^{-n}}\right)^N = \infty .$$

<u>Theorem</u> 3. <u>Let</u> $Z \subseteq \mathbb{R}^N$ <u>and let</u> $\eta \in (*)$ <u>be given</u>. <u>Assume</u>

<u>that, for every</u> $x \in Z$, <u>there exists an</u> $\eta$-<u>section at</u> $x$ <u>which is disjoint with</u> $Z$. <u>Then</u> $|Z| = 0$.

To formulate the appropriate Vitali theorem for this result, we first introduce the relevant Vitali systems. For a fixed $\eta$-function, we define the <u>screened Vitali system</u> $\mathcal{V}_{scr}(\eta)$ by:

$$(A,\mathcal{S}) \in \mathcal{V}_{scr}(\eta) \Rightarrow \forall x \in A \ \exists \delta_0 \ \forall \delta \leq \delta_0 \ \exists \ B = B[y,r] \in \mathcal{S}:$$

$$B \subseteq B[x,\delta]\,,\ r \geq \eta(\delta).$$

<u>Theorem</u> 4. $(A,\mathcal{S}) \in \mathcal{V}_{scr}(\eta)$, $\eta \in (*) \Rightarrow \exists\ (B_n)$ <u>disjoint</u> $\subseteq \mathcal{S}$: $|A \setminus \cup B_n| = 0$.

This result, as well as Theorem 3, is joint work with L. Mejlbro (Math. Ann. 1977). The condition on $\eta$ in both results is best possible.

Clearly, Theorem 4 implies Theorem 3.

Let us reformulate Theorem 3: For $Z \subseteq \mathbb{R}^N$, define the <u>gap-function</u> by

$$\eta(\delta|x) = \sup\{d(y,Z)\,|\,d(y,x) \leq \delta\};\ \delta > 0\,,\ x \in \mathbb{R}^N.$$

Theorem 3 says that if there exists $\eta \in (*)$ such that $\eta(\cdot|x) \geq \eta$ for all $x \in Z$, then $|Z| = 0$.

In case $Z \subseteq \mathbb{R}^1$ then the gap-function has an interpretation as "sunshine functions" related to the function $y \to d(y,Z)$.

Consider the problem of finding a unified Vitali theorem for Lebesgue measure. In more detail, we search for a Vitali system $\mathcal{V}^*$ say, in $\mathbb{R}^N$, purely geometrical in nature, such that:

(a) the Vitali theorem holds for $\mathcal{V}^*$, i.e. $(A, \mathcal{S}) \in \mathcal{V}^* \Rightarrow \exists\ (B_n)$ disjoint $\subseteq \mathcal{S} : |A \setminus \cup B_n| = 0$,

(b) $\mathcal{V}^*$ is of the pointwise type (i.e. not of the uniform type as the screened systems $\mathcal{V}_{scr}(\eta)$ bur rather of a type as the classical system $\mathcal{V}_{hom}$),

(c) $\eta \in (*) \Rightarrow \mathcal{V}_{scr}(\eta) \subseteq \mathcal{V}^*$,

(d) $\mathcal{V}_{hom} \subseteq \mathcal{V}^*$.

En passant, recall that there exists a unified Vitali system which can handle "arbitrary" measures on $\mathbb{R}^N$, via the astounding theorem found independantly by Besicovitch and by A.P. Morse. This Vitali system is simply the system $\mathcal{V}_{hom}(c > 1)$ obtained from $\mathcal{V}_{hom}$ by imposing the extra condition $c > 1$. Thus, one could hope that "all" density theorems, differentiation theorems, disintegration theorems and the like in $\mathbb{R}^N$ can be deduced from results about just two Vitali systems ($\mathcal{V}_{hom}(c > 1)$ and $\mathcal{V}^*$) - and, of course, from theorems outside the scope of Vitali type theorems.

But $\mathcal{V}^*$ is not known. Here is a natural guess: Define $\mathcal{V}^*_{scr}$ by

$$(A, \mathcal{S}) \in \mathcal{V}^*_{scr} \Leftrightarrow \forall x \in A\ \exists \eta \in (*)\ \forall \delta\ \exists\ B = B[y,r] \in \mathcal{S} : B \subseteq B[x,\delta], r \geq \eta(\delta).$$

Does $\vartheta^*_{scr}$ work? Trivially, $\vartheta^*_{scr}$ satisfies (b) and (c). And by a little argument, $\vartheta^*_{scr}$ is also seen to satisfy (d). But we have:

Theorem 5. (M. Talagrand). $\vartheta^*_{scr}$ does not satisfy the crucial condition (a), the Vitali theorem.

We shall not describe the pertinent example here - for one reason, the details are not yet properly collected. Instead, we shall later on give a somewhat weaker example.

## A study of a special case

Let us simplify as much as possible without loosing the flavour of the problem. So we shall aim at geometric criteria for a set to be a Lebesque nullset, and leave the Vitali problem aside. Also, we consider a 1-dimensional space and assume that $Z \subseteq [0,1]$ . Then the complement

$$\Gamma = [0,1] \setminus Z$$

is a disjoint union of open intervals. We shall assume that all these intervals are dyadic intervals.

The usual identification of dyadic intervals with elements of the tree $2^{(\mathbb{N})} = \cup_0^\infty 2^n$ may then be utilized so that the problem is translated into a problem about subtrees of $2^{(\mathbb{N})}$ .

Let $T \subseteq 2^{(\mathbb{N})}$ be a tree. An infinite branch in $T$ we usually denote by $t$ and we often identify $t$ with the corresponding point in $[0,1]$. By $Z$ we denote the set of all infinite branches in $T$ and by $\Gamma$ we denote the complement $\Gamma = [0,1] \setminus Z$. Then $\Gamma$ can also be described as the union of all intervals corresponding to endpoints of $T$. By $\mathrm{end}(T)$ we denote the set of endpoints of $T$. If $\varepsilon \in 2^{(\mathbb{N})} \setminus \{\emptyset\}$, $\varepsilon'$ denotes the neighbour of $\varepsilon$. We assume that $\varepsilon$ and $\varepsilon'$ can not both be endpoints of $T$. For any $\varepsilon \in T$, we denote by $\gamma(\varepsilon)$ the nearest endpoint extending $\varepsilon$ (if there are sevaral such points, we choose the point lying furthest to the right). The relation "$\delta$ extends $\varepsilon$" we write $\delta \geq \varepsilon$. We define $\Delta(\varepsilon)$ and $\Xi(\varepsilon)$ by

$$\Xi(\varepsilon) = \text{lenght of } \gamma(\varepsilon)', \quad \Delta(\varepsilon) = \Xi(\varepsilon) - \text{lenght of } \varepsilon .$$

For any $\gamma \in \mathrm{end}(T)$, we denote by $\alpha(\gamma)$ the point in $T$ of shortest lenght which has $\gamma$ as a nearest endpoint.

The idea is now that for each $t \in Z$, we consider the ball with center $t$ and diameter $2^{-n}$ to be essentially identical with the dyadic interval $t|n$ (of course, this is not quite right so that it needs some adjustments to translate a result for trees into a result of the purely geometric nature which we considered earlier). Our attitude means that $\gamma(t|n)$ replaces the largest ball contained in the ball with center $t$ and diamter $2^{-n}$. As the diameter of $\gamma(t|n)$ is $2^{-\Xi(t|n)}$, we see that the gap-function $\eta$ is now replaced by the function $(t,n) \to 2^{-\Xi(t|n)}$.

The main results of a positive character may be summarized as follows:

Theorem 6. In each of the following cases:

Case $1^o$:

$$\sum_0^\infty 2^{-\Delta(t|n)} = \infty \quad \forall t \in Z,$$

$$\sum_{\gamma \in \text{end}(T)} \Delta(\alpha(\gamma))|\gamma| < \infty,$$

Case $2^o$:

$$\sum_0^\infty \Xi(t|n)^{-1} \, 2^{-\Delta(t|n)} = \infty \quad \forall t \in Z,$$

Case $3^o$:

$$\sum_0^\infty 2^{-\Delta^*(t|n)} = \infty \quad \forall t \in Z, \text{ where } \Delta^*(\varepsilon) = \max(\Delta(\varepsilon), \Delta(\varepsilon')),$$

it can be concluded that $|Z| = 0$.

By Talagrand's example (Theorem 5), the second condition in case $1^o$ can not be dropped.

Note that the result in case $3^o$ is a considerable improvement over the result corresponding to Theorem 3.

All 3 cases are handled by the following result:

<u>Lemma</u>. <u>Assume that there exists a function</u> $\varphi : T \to \mathbb{R}_+$ <u>such that</u>

(a) $$\sum_0^\infty \varphi(t|n)\ 2^{-\Delta(t|n)} = \infty \quad \forall t \in Z$$

<u>and</u>

(b) $$\sum_{\gamma \in \mathrm{end}(T)} \Big( \sum_{\alpha(\gamma) \le \varepsilon < \gamma} \varphi(\varepsilon) \Big) \cdot |\gamma| < \infty$$

<u>hold. Then</u> $|Z| = 0$ .

Actually, condition (a) only comes into play via the following inequality (which only requires (b)):

$$\int_Z \Big( \sum_0^\infty \varphi(t|n)\ 2^{-\Delta(t|n)} \Big) dt < \infty .$$

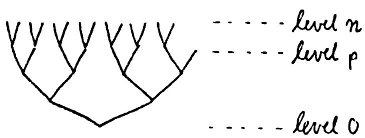
Thus, we may replace case $3^0$ of Theorem 6 by the general result that

$$\int_Z \Big( \sum_0^\infty 2^{-\Delta^*(t|n)} \Big) dt < \infty .$$

As we shall now indicate, $\Delta^*$ can not be replaced by $\Delta$ in this inequality.

Put, for $1 \le p \le n$ ,

$T_{np}$ =




(the figure corresponds to the case $n = 4,\ p = 3$).

Define $T_{n_1,p_1} \otimes T_{n_2,p_2}$ by taking $T_{n_2,p_2}$ and placing this tree on top of all the $2^{n_1} - 2^{(n_1-p_1)}$ end-points of $T_{n_1, p_1}$ of length n. This process may be continued to obtain infinite "tensor products" of trees.

<u>Theorem</u> 7. <u>Assume that</u> $p_\nu$ <u>and</u> $n_\nu$ <u>are numbers such that</u> $1 \leq p_\nu \leq n_\nu$ ; $\nu \geq 1$ <u>and such that</u>

$$\sum_1^\infty 2^{-p_\nu} < \infty , \quad \sum_1^\infty p_\nu 2^{-p_\nu} = \infty .$$

<u>Put</u> $T = T_{n_1 p_1} \otimes T_{n_2 p_2} \otimes \cdots$ . <u>Then</u>

$$\int_Z \left( \sum_0^\infty 2^{-\Delta(t|n)} \right) dt = \infty .$$

We leave the details of proof to the reader.

Remark that the above example (examples) can not be used to obtain an example as strong as Talagrand's. The reason is that for all the examples, it can be shown that $|Z_\infty| = 0$ where

$$Z_\infty = \left\{ t \in Z \,\middle|\, \sum_0^\infty 2^{-\Delta(t|n)} = \infty \right\}.$$