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We shall try to mention some interrelations between the disjoint refinement property and some other properties of Boolean algebras or of compact topological spaces. The aim of the present paper is to give a spirit of a global picture rather than the most general formulations or the precise proofs, which will appear – or have appeared – elsewhere. The language of the paper is a mixture of Boolean and topological terms: the reader is requested to keep in mind the Stone duality during the reading.

Let us start with the basic notion. If $\mathcal{B}$ is a Boolean algebra, κ cardinal, $\mathcal{A} = \{a_\alpha : \alpha < \kappa\}$ a family of non-zero members of $\mathcal{B}$, then the family $\{d_\alpha : \alpha < \kappa\}$ of non-zero members is a disjoint refinement of $\mathcal{A}$, if $d_\alpha \wedge d_\beta = \mathbb{0}$ for each $\alpha < \beta < \kappa$ and $d_\alpha \leq a_\alpha$ for each $\alpha < \kappa$. Of course, the notion of the disjoint refinement is meaningful not only for Boolean algebras – one may ask e.g. whether some given family of open sets in a topological space has a disjoint refinement by open subsets as well.

If X is a topological space, $x \in X$ and κ cardinal, call a point x to be a κ-point, if there is a family $\{U_\alpha : \alpha < \kappa\}$ of pairwise disjoint open subsets of X with $x \in \overline{U}_\alpha$ for each $\alpha < \kappa$. We remind Comfort and Hindman's observation:

<u>Let X be a topological space, $x \in X$, $\chi(x) \leq \kappa$ and sup-</u>

pose that each non-void open set in X contains at least κ non-void pairwise disjoint open subsets. Then x is a κ-point if and only if some neighborhood base of x has a disjoint refinement.

Indeed, if x is a κ-point, choose $\mathcal{B} = \{B_\alpha : \alpha < \kappa\}$ an arbitrary neighborhood base of x, let $\{U_\alpha : \alpha < \kappa\}$ be a pairwise disjoint collection of open sets with $x \in \overline{U}_\alpha$ for each $\alpha < \kappa$. Then $\{U_\alpha \cap B_\alpha : \alpha < \kappa\}$ is the desired disjoint refinement. If $\{D_\alpha : \alpha < \kappa\}$ is the disjoint refinement of some neighborhood base $\{B_\alpha : \alpha < \kappa\}$ of the point x, choose a pairwise disjoint family $\{D_{\alpha\xi} : \xi < \kappa\}$ for each $\alpha < \kappa$, where $D_{\alpha\xi}$ is open, $\emptyset \neq D_{\alpha\xi} \subset D_\alpha$. For $\xi < \kappa$, define U_ξ by the rule $U_\xi = \bigcup\{D_{\alpha\xi} : \alpha < \kappa\}$.

The notion of a κ-point may be generalized or rephrased by various ways. Call a set M in a topological space X to be a κ-set, if there is a family of pairwise disjoint open subsets $\{U_\alpha : \alpha < \kappa\}$ with $M \subset \overline{U}_\alpha$ for every $\alpha < \kappa$.

Call a family $\mathcal{A}$ of non-zero members of a Boolean algebra $\mathcal{B}$ (of non-empty open subsets of a topological space X) to be strictly κ-decomposable, if there is a family $\{U_\alpha : \alpha < \kappa\}$ of pairwise disjoint non-zero members of $\mathcal{B}$ (non-empty open subsets of X) such that $A \cap U_\alpha \neq 0$ (is non-void) for each $A \in \mathcal{A}$ and each $\alpha < \kappa$. (There is a commonly used notion of κ-decomposability due to Chang [Ch], our definition is a strengthening of it.) Clearly a point $x \in X$ is a κ-point iff its neighborhood system is strictly κ-decomposable; a set $M \subset X$ is a κ-set iff a family $\{B \subset X : B$ is open and $B \cap M \neq \emptyset\}$ is strictly κ-decomposable.

Call a point $x \in X$ to be a semi-κ-point if there is a family $\{U_\alpha : \alpha < \kappa\}$ of non-void open subsets of X such that $x \in \bar{U}_\alpha$ for each $\alpha < \kappa$, but $x \notin \overline{U_\alpha \cap U_\beta}$ whenever $\alpha < \beta < \kappa$. Clearly every κ-point is a semi-κ-point, but not vice versa. For instance, each $x \in R$ (= the real line) is a semi-2^ω-point, though it cannot be (more than ω)-point.

To demonstrate the basic techniques and reasoning used in this area of problems, let us prove that each point $x \in \beta(N) - N$ is an ω_1-point. The fact is not as interesting as a proof of it, since we shall show later that more is true.

The crucial observation is as follows: Suppose $\{X_\alpha : \alpha \leq \omega\}$ to be a set of non-void clopen subsets of $\beta(N) - N$ such that $X_\alpha \supset X_\beta$ for each $\alpha < \beta \leq \omega$. If some clopen set $M \subset \beta(N) - N$ meets $X_\alpha - X_{\alpha+1}$ for infinitely many α's, then $M \cap \operatorname{Int} \bigcap \{X_\alpha : \alpha < \omega\} - X_\omega$ is non-empty. Indeed, let Y_α, K be subsets of N representing X_α, M. Choose some $x_\alpha \in K \cap \bigcap \{Y_\gamma : \gamma \leq \alpha\} - (Y_{\alpha+1} \cup Y_\omega)$ if possible; from the assumptions we know that we can find an infinite set L of those x_α's. Now $\emptyset \neq \operatorname{cl}_{\beta(N)} L - L \subset M \cap \operatorname{Int} \bigcap \{X_\alpha : \alpha < \omega\} - X_\omega$. It should be noted that replacing ω by κ anywhere (κ regular), we obtain an analogous observation for $U(\kappa)$, the space of all uniform ultrafilters on κ.

Now let $x \in \beta(N) - N$. There is a regular uncountable cardinal λ and a family of clopen subsets $\{X_\alpha : \alpha < \lambda\}$ with the following properties: For every $\alpha < \beta < \lambda$, $X_\alpha \supset X_\beta \ni x$ and $x \in \operatorname{bd} \bigcap \{X_\alpha : \alpha < \lambda\}$. If x is a P-point, then such a family can be found easily, so suppose it is not: There is a partition $\{A_n : n < \omega\}$ of N such that $|\{n < \omega : |M \cap A_n| = \omega\}| = \omega$ for each $M \in x$. Thus if f is a mapping from ω to ω and if X_f denotes the set $\{m \in N : \text{if } m \in A_n\text{, then}$

$m > f(n)\}$, then $X_f \in x$. Order ${}^{\omega}\omega$ by $f < g$ if $f(n) \geq g(n)$ for finitely many n's only; then there is some uncountable regular λ and a set $\{f_\alpha : \alpha < \lambda\}$ which has no upper bound in ${}^{\omega}\omega$, moreover, $\alpha < \beta < \lambda$ implies $f_\alpha < f_\beta$ and all functions are increasing. It remains to define $X_\alpha = cl_{\beta(N)} X_{f_\alpha} - N$.

According to our starting observation, the set $\{\alpha < \lambda : U \cap (\mathrm{Int} \bigcap \{X_\beta : \beta < \alpha\} - X_\alpha) \neq \emptyset$ is ω-closed and unbounded in λ for each clopen U containing x. By Fodor's theorem, the set λ can be decomposed into pairwise disjoint sets $\{Z_\xi : \xi < \lambda\}$ such that each Z_ξ meets each ω-closed unbounded subset of λ. Now, define $U_\xi = \bigcup\{\mathrm{Int} \bigcap \{X_\beta : \beta < \alpha\} - X_\alpha : \alpha \in Z_\xi\}$. Clearly $x \in \overline{U}_\xi$ for each $\xi < \lambda$, U_ξ is non-empty open and $U_\xi \cap U_\eta = \emptyset$, which was to be proved.

Analysing the proof just given, one can obtain more information than stated.

At first, let $\{A_n : n < \omega\}$ be a pairwise disjoint family of non-void clopen subsets of $\beta(N) - N$. Then the family $\{M \subset \beta(N) - N : M$ is clopen and $|\{n < \omega : M \cap A_n \neq \emptyset\}| = \omega\}$ is strictly ω_1-decomposable (in fact, strictly λ-decomposable).

Second, there is no reason for dealing with ω only. Let κ be a regular cardinal, let $\{A_\alpha : \alpha < \kappa\}$ be a pairwise disjoint family of non-void clopen subsets of $U(\kappa)$. Then the family $\{M \subset U(\kappa) : M$ is clopen and $|\{\alpha < \kappa : M \cap A_\alpha \neq \emptyset\}| = \kappa\}$ is strictly κ^+-decomposable (in fact, strictly λ_κ-decomposable, if one defines λ_κ for ${}^{\kappa}\kappa$ similarly as λ was defined for ${}^{\omega}\omega$).

Third, let $\mathcal{U}$ be a family of clopen subsets of $\beta(N) - N$. Suppose that there is a regular λ and a family $\{X_\alpha : \alpha < \lambda\}$

of clopen subsets of $\beta(N) - N$ satisfying the following: $X_\alpha \supset X_\beta$ for $\alpha < \beta < \lambda$, $U \cap X_\alpha \neq \emptyset$ for each $U \in \mathcal{U}$ and each $\alpha < \lambda$, $U - \bigcap \{X_\alpha : \alpha < \lambda\} \neq \emptyset$ for each $U \in \mathcal{U}$. Call such a family a tower for $\mathcal{U}$. Then $\mathcal{U}$ is strictly λ-decomposable.

As in the previous cases, we can work in $U(\kappa)$ as well. If κ is a regular uncountable cardinal, then the family $\{\mathrm{cl}_{\beta(\kappa)} S \cap U(\kappa) : S$ is a stationary subset of $\kappa\}$ is strictly κ^+-decomposable. To show this, notice that the family of closed unbounded sets in κ is closed under the diagonal intersections, hence there is a tower for the given system in $U(\kappa)$, the length of which is at least κ^+. As a corollary we deduce that for regular uncountable κ, each ultrafilter extending the filter of closed unbounded sets is a κ^+-point in $U(\kappa)$; in fact the same holds for an arbitrary $x \in U(\kappa)$, but we omit the proof here.

Fourth, each point $x \in \beta(N) - N$ is a 2^ω-point. We know that there is a tower $\{X_\alpha : \alpha < \lambda\}$ for x in $\beta(N) - N$. Every set $Q_\alpha = \bigcap \{X_\beta : \beta < \alpha\} - X_\alpha$ is closed G_δ whenever $\mathrm{cf}(\alpha) = \omega$. Using essentially the same argument which enabled us to prove that there is a tower for non-P-point, we can find a tower $\{X^\alpha_\beta : \beta < \lambda\}$ with $X^\alpha_\beta \subset Q_\alpha$, denote $Q_{\alpha\beta} = \bigcap \{X^\alpha_\gamma : \gamma < \beta\} - X^\alpha_\beta$ for β with countable cofinality. Repeating the reasoning further, we shall find for each $k < \omega$ and $\alpha_1, \alpha_2, \dots, \alpha_k$ with $\mathrm{cf}(\alpha_i) = \omega$, $\alpha_i < \lambda$, a closed G_δ-set $Q_{\alpha_1\alpha_2\dots\alpha_k}$ such that $Q_{\alpha_1\alpha_2\dots\alpha_k} \subset Q_{\alpha_1\alpha_2\dots\alpha_{k-1}}$, moreover, if M is clopen in $\beta(N) - N$ and $M \cap \bigcap \mathrm{bd}\, Q_{\alpha_1\alpha_2\dots\alpha_{k-1}} \neq \emptyset$, then M meets $\mathrm{Int}\, Q_{\alpha_1\alpha_2\dots\alpha_k}$ for

λ-many α_k's . Next, if $s = (\alpha_1, \alpha_2, \ldots)$ is an infinite sequence of ordinals α_i with $\alpha_i < \lambda$, $cf(\alpha_i) = \omega$, there is an open set $P_s = \mathrm{Int} \bigcap \{Q_{s \restriction n} : n < \omega\}$. A straightforward branching argument shows that each clopen neighborhood U of x meets 2^{ω} sets P_s .

It remains to notice that the following trivially holds: If $\mathcal{B}$ is a Boolean algebra, κ an infinite cardinal, $\{a_\alpha : \alpha < \kappa\}$ a set of non-zero members of $\mathcal{B}$ and $\mathcal{P}$ a pairwise disjoint family of non-zero members of $\mathcal{B}$ such that $|\{p \in \mathcal{P} : a_\alpha \wedge p \neq 0\}| \geq \kappa$ for each $\alpha < \kappa$, then $\{a_\alpha : \alpha < \kappa\}$ has a disjoint refinement. So when noting that in $\beta(N) - N$, each family $\mathcal{A}$ of clopen sets has a disjoint refinement if and only if $\mathcal{A}$ is strictly 2^{ω}-decomposable, the proof will be complete then.

Fifth, we have proved a bit more again: Let $\{A_n : n < \omega\}$ be a pairwise disjoint family of clopen sets in $\beta(N) - N$. Then the family $\{U \subset \beta(N) - N : U$ is clopen and $|\{n < \omega : U \cap A_n \neq \emptyset\}| = \omega\}$ has a disjoint refinement, in the other words, it is strictly 2^{ω}-decomposable, or equivalently, $\mathrm{bd} \bigcup \{A_n : n < \omega\}$ is a 2^{ω}-set.

This leads to the questions [He] : Is each nowhere dense subset of $\beta(N) - N$ a 2^{ω}-set ? Which families of clopen subsets of $\beta(N) - N$ are strictly 2^{ω}-decomposable ?

Up to now we know the following (as usually adopted M^* denotes the set $\mathrm{cl}_{\beta(N)} M - N$) :

(a) Let $\{A_n : n < \omega\}$ be a partition of N into finite sets with $\sup |A_n| = \omega$. Then the family $\{M^* : \sup |M \cap A_n| = \omega\}$ is strictly 2^{ω}-decomposable.

(b) Suppose there is a maximal disjoint collection of clopen sets $\mathcal{A}$ in $\beta(N) - N$ of cardinality ω_1 . Then $N^* - \bigcup \mathcal{A}$

is a 2^{ω}-set in $\beta(N) - N$.

(c) Let f be a bijection of N onto Q (=the rational numbers). Then $\{M^* : f[M]$ has infinitely many accumulation points in $R\}$ is strictly 2^{ω}-decomposable.

(d) Let κ be a cardinal, $\kappa \leq 2^{\omega}$. Then there exists an almost disjoint family $\mathcal{A}$ of countable subsets of κ such that each uncountable subset of κ contains some member of $\mathcal{A}$.

Now, let us turn our attention to semi-κ-points. Suppose X to be a compact Hausdorff space, $G(X)$ its absolute, $\pi :$ $G(X) \longrightarrow X$ the projection. If $x \in X$ is a semi-κ-point, then the subspace $\pi^{-1}(x)$ of $G(X)$ has cellularity at least κ. Indeed, if $\{U_{\alpha} : \alpha < \kappa\}$ witnesses for x being a semi-κ-point, then $\pi^{-1}[U_{\alpha}]$ $(= \{p \in G(X) : \pi(p) \in \overline{U}_{\alpha}\})$ are open subsets of $G(X)$ and $\pi^{-1}[U_{\alpha}] \cap \pi^{-1}[U_{\beta}] \cap \pi^{-1}(x) = \emptyset$ but $\pi^{-1}[U_{\alpha}] \cap \pi^{-1}(x) \neq \emptyset$.

We shall show that <u>under CH, each point in $\beta(N) - N$ is a semi-2^c-point.</u> Enumerate as $\{U_{\alpha} : \alpha < \omega_1\}$ a clopen neighborhood basis of x and choose a tower $\{X_{\xi} : \xi < \omega_1\}$ for x. An easy induction gives us a pairwise disjoint system $\{A_{\alpha} : \alpha < \omega_1\}$ of clopen subsets of $\beta(N) - N$ and a set $\{\xi(\alpha) : \alpha < \omega_1\}$ of ordinals with the following properties: if $\alpha < \beta < \omega_1$, then $U_{\alpha} \supset A_{\beta}$, $\xi(\alpha) < \xi(\beta)$ and $A_{\alpha} \cap$ $\cap X_{\xi(\beta)} = \emptyset$. Choose a maximal almost disjoint family $\{Z_{\eta} :$ $\eta < 2^c\}$ of uncountable subsets of ω_1. Then $\{V_{\eta} =$ $= \bigcup\{A_{\beta} : \beta \in Z_{\eta}\} : \eta < 2^c\}$ is the system of open subsets of $\beta(N) - N$ with $x \in \overline{V}_{\eta}$ for each η, $x \notin \overline{V_{\eta} \cap V_{\eta'}}$ for $\eta \neq \eta'$.

Up to now we have dealt with particular examples of Boolean algebras. Let us briefly consider the general case.

follows from the fact that each extremally disconnected compact space X can be mapped onto $2^{w(X)}$. Needless to say that the proof of the last result is yet another example of a careful examination of the disjoint refinement properties.

The list of references given below is far from being complete. A more comprehensive set of titles may be found e.g. in the paper [BSV_2].

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