Robert Frič; Miroslav Hušek

Epireflective subcategories of convergence spaces

In: Zdeněk Frolík (ed.): Abstracta. 8th Winter School on Abstract Analysis. Czechoslovak Academy of Sciences, Praha, 1980. pp. 68–72.

Persistent URL: http://dml.cz/dmlcz/701179

EIGHTH WINTER SCHOOL (1980)

EPIREFLECTIVE SUBCATEGORIES OF CONVERGENCE SPACES

R. Frič and M. Hušek

Although the category Conv of convergence spaces (objects are closure spaces in which the closure operator (not necessarily idempotent) is induced by means of converging sequences (we always suppose unique sequential limits), see [8], [9], [2], [4], and morphisms are sequentially continuous mappings) seems to be very similar to the category Cl of closure spaces (or Top of topological spaces); looking at special properties one can find important differences. We shall look here at epireflective subcategories (always full subcategories) of Conv, mainly generated by "classical" spaces as subspaces of the real line and the plane.

First we shall recall general results for epireflective subcategories (see [6]) transformed to Conv.

1. A subclass of Conv is bireflective iff it is productive and hereditary.

2. A subclass of Conv is epireflective iff it is productive and closed hereditary.

3. Every class $\mathcal{E}$ in Conv is contained in the smallest bireflective class $\mathcal{E}_b = \{X \mid X$ is isomorphic to a subspace of $\prod E_i$, $E_i \in \mathcal{E}\}$ and in the smallest epireflective class $\mathcal{E}_e = \{X \mid X$ is isomorphic to a closed subspace of $\prod E_i$, $E_i \in \mathcal{E}\}$. Note that the closure operator in $\prod E_i$ is induced by the pointwise convergence of sequences. Clearly, $\mathcal{E}_e \subset \mathcal{E}_b$, $\mathcal{E}_b = (\mathcal{E}_e)_b$.

The elements of $\mathcal{E}_b$ are called _$\mathcal{E}$-sequentially regular spaces_ and the elements of $\mathcal{E}_e$ are called _$\mathcal{E}$-sequentially complete spaces._ If $\mathcal{E}$ consists of a single space E, then $\mathcal{E}$ is replaced by E.

Theorem 1. Let $X \in \text{Conv}$. Then the following are equivalent:
(i) $X \in \mathcal{E}_b$.
(ii) A sequence $\langle x_n \rangle$ converges to a point x in X iff the sequence $\langle fx_n \rangle$ converges to fx in E for each morphism $f: X \to E$, $E \in \mathcal{E}$.

Theorem 2. Let $X \in \text{Conv}$. Then the following are equivalent:
(i) $X \in \mathcal{E}_e$.
(ii) A sequence $\langle x_n \rangle$ converges in X iff the sequence $\langle fx_n \rangle$ converges in E for each morphism $f: X \to E$, $E \in \mathcal{E}$.

Conditions (ii) in the above theorems with $\mathcal{E} = \{R\}$ were originally taken as definitions of sequential regularity (cf. [8]) and sequential completeness (cf. [4]), and allow generalizations of these notions for various classes of maps, e.g., measures ([9]), sets of functions ([4]).

It can be proved that if $X \in \mathcal{E}_b$ and $f: X \to Y$ is the epireflection of X in $\mathcal{E}_e$, then f is an embedding which can be described as follows.

Theorem 3. Let $X \in \mathcal{E}_b$. If we take the epireflection Z of X in $\mathcal{E}$-compact spaces in Cl (i.e., homeomorphs of closed subspaces of products from $\mathcal{E}$), then the smallest sequentially closed subset of Z containing X and endowed with the sequential closure is the epireflection of X in $\mathcal{E}_e$.

Thus, e.g., if E is a metrizable space, then the epireflection of X in E_e can be obtained via the E-compactification $\beta_E X$ of X (hence via βX or υX if $E = I$ or $E = R$, cf. [7], [1], [2]).

Clearly, the class $\mathcal{S}$ of all singletons is the smallest bireflective or epireflective class in Conv; this class is not interesting.

Denote by D the discrete two-point space.

Observation. There is no bireflective class in Conv strictly in between $\mathcal{S}$ and D_b. There is no epireflective class in Conv strictly in between $\mathcal{S}$ and D_e. Clearly, $\mathcal{E}_b = D_b$ iff $\mathcal{E}$ consists of D-sequentially regular spaces and $\mathcal{E}$ contains at least one nondegenerate space.

If $E \subset R$, then $E \in D_b$ iff E is 0-dimensional in the topological sense, i.e., E contains no nondegenerate interval. If E contains such an interval, then it follows from [2] that E-sequential regularity does not depend on the form of the interval (i.e., whether it is closed, open or half-open). Also, fpr $E \subset R$ we have $E \in D_e$ iff $E \in D_b$. Thus we have the following result.

Theorem 4. Let $E \subset R$, card $E > 1$. Then E_b coincides either with D_b or R_b and, similarly, E_e coincides either with D_e or R_e.

Thus, in Conv the situation is different from the situation in Cl or Top, where subspaces D, N, I, $\bigcup\{[2n, 2n+1]\}$ and R (and no other) of the real line generate nontrivial different epireflective classes. For bireflective classes, of course, the situation is the same as in Conv, i.e., only two nontrivial different classes (generated by D and R) exist.

We can express the above situation by means of the following relations: we write $E \mu F$ if $E_b = F_b$, resp. $E \rho F$ if $E_e = F_e$. Clearly, $E \rho F$ implies $E \mu F$. Then Theorem 4 asserts that $E,F \subset R$ and $E \mu F$ imply $E \rho F$.

Note (cf. [3]) that there is a D-sequentially regular space such that its epireflection in R_e is not D-sequentially regular, i.e., the situation here is similar to that in Top.

In Top it was proved first by P. Nyikos that realcompact 0-dimensional spaces do not coincide with N-compact spaces. In Conv the corresponding problem is still open.

Problem 1. Does there exist a D-sequentially regular space which is R-sequentially complete but fails to be D-sequentially complete?

Let X be a realcompact 0-dimensional space. Then (cf. [2]) the associated convergence space (i.e., the set X equipped with the closure operator induced by means of convergent sequences in the topological space X) is D-sequentially regular and R-sequentially complete. However, to solve the above problem, one cannot use spaces similar to those used by P. Nyikos ([10]) or E. Pol ([11]) because of the following fact.

Theorem 5. Let X be a sequential realcompact 0-dimensional topological space. Then the associated convergence space is D-sequentially complete.

Proof. Suppose that a sequence $\langle x_n \rangle$ does not converge in X. If $\langle x_n \rangle$ has more than one accumulation point, then any two of these accumulation points can be separated by a continuous function f: X ⟶ D. Thus the sequence $\langle fx_n \rangle$ does not converge in D. If $\langle x_n \rangle$ has at most one accumulation point, then there is a subsequence $\langle y_n \rangle$ of $\langle x_n \rangle$ such that $\bigcup (y_n)$ is a closed discrete infinite subset of X. Then there is a continuous function g: X ⟶ R such that the sequence $\langle gy_n \rangle$ is unbounded (otherwise $\bigcup (y_n)$ would be compact) and hence there is a subsequence $\langle z_n \rangle$ of $\langle y_n \rangle$ which is uniformly discrete. Consequently, subsequences $\langle z_{2n} \rangle$ and $\langle z_{2n+1} \rangle$ of $\langle x_n \rangle$ can be separated by a continuous function f: X ⟶ D and the sequence $\langle fx_n \rangle$ does not converge in D.

The situation for subspaces of the plane is different; they generate infinitely many different epireflective classes.

Theorem 6. There are subspaces F_α of R^2, $\alpha \in 2^{2^\omega}$, such that F_α is not F_β-sequentially regular for $\alpha \neq \beta$.

Proof. One can take for F_α the spaces constructed in [5].

Thus we have 2^{2^ω} incomparable epireflective classes generated by subspaces of the plane. The same spaces F_α from [5] can be used to show that, unlike in R, there are spaces $E, F \subset R^2$ such that $E \rho F$, E nonς F.

References

[1] R. Frič: Sequential envelope and subspaces of the Čech-Stone compactification. General Topology and its Relations to Modern Analysis and Algebra III, Proc. Third Prague Topological Sympos., 1971. Academia, Praha, 1972, 123-126.

[2] R. Frič: On E-sequentially regular spaces. Czechoslovak Math. J. 26(101) (1976), 604-612.

[3] R. Frič and M. Hušek: On projectively generated spaces. Comment. Math. Univ. Carolinae 20(1979), 194.

[4] R. Frič and V. Koutník: Sequentially complete spaces. Czechoslovak Math. J. 29(104)(1979), 287-297.

[5] J. de Groot: Groups represented by homeomorphism groups, I. Math. Ann. 138(1959), 80-102.

[6] H. Herrlich and G. Strecker: Category theory. Allyn and Bacon Inc., Boston 1973.

[7] V. Koutník: On sequentially regular convergence spaces. Czechoslovak Math. J. 17(92)(1967), 232-247.

[8] J. Novák: On convergence spaces and their sequential envelopes. Czechoslovak Math. J. 15(90)(1965), 74-100.

[9] J. Novák: On sequential envelopes defined by means of certain classes of continuous functions. Czechoslovak Math. J. 18(93)(1968), 450-465.

[10] P. Nyikos: Not every 0-dimensional realcompact space is N-compact. Bull. Amer. Math. Soc. 77(1972), 392-396.

[11] E. Pol: Remark about Juhász - Kunen - Rudin construction of a hereditary separable non-Lindelöf space. Bull. Acad. Pol. Sci. 24(1976), 749-751.