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Weak-join of matriods

Terms of use:

Weak-join of matroids

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This note deals with glueing of matroids. Some construc-
tions in the matroid theory can be considered as glueing, e.g.
the sum of matroids, the simultaneous extension [2], the
Dilworth truncation [3]. The presented approach aims to
applications in Ramsey theory, for particular results see [4].
We introduce the notion of the weak-join which, if it exists,
is the free-est amalgamation. We give a sufficient condition
for the existence of a weak-join.

A <u>matroid</u> $M(X)$ is a set X with a rank function
$r_M \colon \mathcal{P}(X) \longrightarrow N$ satisfying:

1/ $r(\emptyset) = 0$

2/ $r(\{x\}) \leq 1$ for $x \in X$

3/ $r(A) \leq r(B)$ for $A \subseteq B \subseteq X$

4/ $r(A) + r(B) \geq r(A \cup B) + r(A \cap B)$ for $A, B \subseteq X$

A matroid M is called modular if

$r(F) + r(G) = r(F \cup G) + r(F \cap G)$ for every pair F, G of flats.

For $A \subseteq X$ the restriction of M to the subset A is denoted
by $M|A$.

<u>Definition 1</u>: Let $\mathcal{H} = (V, (E_i \mid i \in I))$ be a hypergraph and
$\mathcal{M} = (M_i(E_i) \mid i \in I)$ be a system of matroids. A matroid $M(V)$
is called an <u>amalgamation of $\mathcal{M}$ with respect to $\mathcal{H}$</u> if
$M \mid E_i = M_i(E_i)$.

A. Weak-join of two matroids

If $M(X_1 \cup X_2)$ is an amalgamation of matroids $M_1(X_1)$, $M_2(X_2)$ then

$$r_M(A) \leq r_1(B \cap X_1) + r_2(B \cap X_2) - r_1(B \cap X_1 \cap X_2)$$

for every $A \subseteq B \subseteq X_1 \cup X_2$.

Definition 2: Let $M_1(X_1)$, $M_2(X_2)$ be matroids with $M_1|X_1 \cap X_2 = M_2|X_1 \cap X_2$. Put

$$\rho(A) = r(A \cap X_1) + r(A \cap X_2) - r(A \cap X_1 \cap X_2) \quad \text{and}$$

$$R(A) = \min\{\rho(B) \mid A \subseteq B\} \quad \text{for} \quad A \subseteq X_1 \cup X_2.$$

If R is a rank function then the matroid $M(X_1 \cup X_2)$ defined by R is called a weak-join of M_1 and M_2.

Theorem 1: Let $M_1(X_1)$, $M_2(X_2)$ be matroids with $M_1|X_1 \cap X_2 = M_2|X_1 \cap X_2$. Then

(i) If a weak-join exists then it is an amalgamation of M_1 and M_2.

(ii) If a weak-join $M(X_1 \cup X_2)$ exists and $N(X_1 \cup X_2)$ is another amalgamation then the identity mapping $i: M \longrightarrow N$ is a weak map, i.e. weak-join is the free-est amalgamation (with respect to weak maps).

(iii) If the matroid $M_1|X_1 \cap X_2$ is modular then the weak-join of M_1 and M_2 exists.

Example: The smallest non-modular matroid C_4 is that formed by four points in general position in the plane. The following picture gives an example of two matroids which intersect in C_4 and which have no amalgamation.
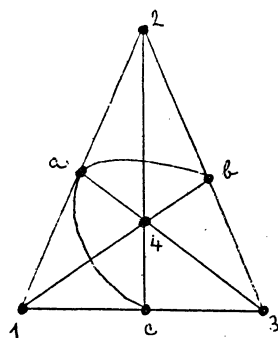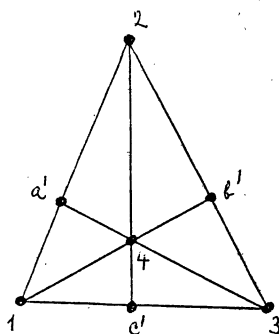
Fano   Fano⁻

In [1] Brylawski introduced a notion of a strong-join,
which is, if it exists, the free-est amalgamation with respect
to strong maps. Let us remark that any strong-join is the
weak-join.

B. Weak-join with respect to tree

<u>Definition 3</u>: A hypergraph $(V,(E_i \mid i \in I))$ is called a <u>tree-</u>
<u>hypergraph</u> if there exists a graph $\mathcal{T} = (I,T)$ which is a tree
such that: if the vertex i lies on the path between vertices
j and k in $\mathcal{T}$ then $E_j \cap E_k \subseteq E_i$.

The following definition is a generalization of the weak-join
to the tree-hypergraph.

<u>Definition 4</u>: Let $\mathcal{K} = (V,(E_i \mid i \in I))$ be a tree-hypergraph
and $\mathcal{M} = (M_i(E_i) \mid i \in I)$ be a system of matroids satisfying
$M_i | E_i \cap E_j = M_j | E_i \cap E_j$ for every $i,j \in I$. Put

$$\varrho(A) = \sum_{i \in I} r(A \cap E_i) - \sum_{i \neq j \in I} r(A \cap E_i \cap E_j) + \ldots + (-1)^{|I|+1} r\left(A \cap \bigcap_{i \in I} E_i\right)$$
$$R(A) = \min \{ \varrho(B) \mid A \subseteq B \} \quad \text{for} \quad A \subseteq V.$$

If R is a rank function then the matroid $M(V)$ defined by R
is called a weak-join of $\mathcal{M}$ with respect to $\mathcal{K}$.

Theorem 2: Let $\mathcal{K}$ and $\mathcal{M}$ be as in the above definition. Then

(i) If the weak-join of $\mathcal{M}$ with respect to $\mathcal{K}$ exists then
     it is the free-est amalgamation.

(ii) If $M_i | E_i \cap E_j$ is modular for every $i, j \in I,\ i \neq j$ then
     the weak-join of $\mathcal{M}$ with respect to $\mathcal{K}$ exists.

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