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The Hales-Jewett class [A] is defined as follows:

Definition: Let A be a finite set, $k \le n$ be non-negative integers. $[A]\binom{n}{k}$ is then the set of mappings $f: n = \{0,\dots,n-1\} \to A \cup \{\lambda_0,\dots,\lambda_{k-1}\}$ satisfying (1) $f^{-1}(\lambda_i) \neq \emptyset$ for all $i < k$ and (2) $\min f^{-1}(\lambda_i) < \min f^{-1}(\lambda_j)$ for all $i < j < k$.

Parameter-words $f \in [A]\binom{n}{m}$ and $g \in [A]\binom{m}{k}$ may be composed yielding $f \cdot g \in [A]\binom{n}{k}$, where $f \cdot g(i) = f(i)$ for $f(i) \in A$ and $f \cdot g(i) = g(j)$ for $f(i) = \lambda_j$.

Motivation: $f \in [A]\binom{n}{k}$ is the set of embeddings of the k-dimensional cube A^k into A^n, where the subcube described by f is given by $\{fg \mid g \in [A]\binom{k}{0}\} \subseteq A^n$. The following partition theorem which is due to Hales and Jewett [2] in the case $k = 0$ and due to Graham and Rothschild [1] for $k > 0$ is well-known and is in fact one of the major tools for partition (Ramsey) theory:

Theorem: Let A be a finite set. $\forall\, \delta, m, k\ \exists n: n \xrightarrow{[A]} (m)^k_\delta$, i.e. for every coloring $\Delta: [A]\binom{n}{k} \to \delta$ there exists $f \in [A]\binom{n}{m}$ such that $\Delta_f: [A]\binom{m}{k} \to \delta$ defined by $\Delta_f(g) = \Delta(f \cdot g)$ is a constant mapping.

We can prove the following strengthening for $k = 0$:

<u>Theorem:</u> Let A be a finite alphabet. $\forall\ \delta, m, g\ \exists\ S \subset [A]\binom{n}{m}$ such that α $S \xrightarrow{[A]} (m)^{0}_{\delta}$, (i.e. for every mapping $\Delta : [A]\binom{n}{0} \to \delta$ there exists an $f \in S$ such that Δ_f is constant) and β the hypergraph $H^{0}(S)$ with vertex-set $[A]\binom{n}{0}$ and ege set $\{\{fg \mid g \in [A]\binom{m}{0}\} \mid f \in S\}$ has girth at least g (i.e. the m-subcubes in S do no form short cycles).

For $k = 1$ we only have a result for $A = \{0\}$:

<u>Theorem:</u> $\forall\ \delta, m, g\ \exists\ S \subset [\{0\}]\binom{n}{m}$ such that ⓐ $S \xrightarrow{[\{0\}]} (m)^{1}_{\delta}$ and ⓑ $H^{1}(S)$ has girth at least g .

These two theorems have many interesting corollaries, here we state only a few of them:

<u>Corollary 1:</u> Let F be a finite field and let δ, m, g be non-negative integers. Then there exists a family of affine m-dimensional subspaces of an n-dimensional affine space over F such that ⓐ for every coloring of the affine points in F^n with δ many colors there exists an m-dimensional subspace in S with all its affine points colored the same and ⓑ the m-dimensional spaces in S do not form cycles shorter than g .

<u>Corollary 2:</u> Let F be a finite field and let δ, m, g be non-negative integers. Then there exists a family of m-dimensional homogeneous subspaces of the n-dimensional vector space over F (for some sufficiently large n) such that ⓐ for every coloring of the 1-dimensional homogeneous subspaces of the n-dimensional vector space with δ many colors there exists an m-dimensional homogenenous

subspace in S with all its 1-dimensional subspaces colored the same and (β) the m-dimensional subspaces in S do not form cycles shorter than g (with respect to intersection in 1-dimensional subspaces).

<u>Corollary 3:</u> Let $A\,\underline{x} = 0$ be a homogeneous partition regular system of equations (see [3]). Then for every pair δ,g of non-negative integers there exists a family S of solutions of $A\,\underline{x} = 0$ such that (α) for every coloring $\Delta: \cup S \to \delta$ there exists a monochromatic solution in S and (β) the hypergraph S does not contain cycles shorter than g. This generalizes a result of Spencer's [4] for arithmetic progression, moreover, we have a constructive proof for it. Details will appear elsewhere.

<u>References</u>

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