BANACH-STONE THEOREMS FOR NON-SEPARABLY VALUED BOCHNER L^∞-SPACES

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1. INTRODUCTION

In the author's talk at the conference an example has been given for the fact that a plausible seeming description of the extremal points in a Bochner space $L^p(\mu,V)$, in terms of their values, that is valid for separable V, cannot be generalized to *non-separable* spaces. An essential tool for this construction was the Stonean space of μ's measure algebra. Meanwhile this example has been published elsewhere [4].

One of the goals of this article is to give a *positive* result for non-separable spaces in a similar problem (relating geometric properties of $L^\infty(\mu,V)$ to those of V). In [2] Cambern has shown a Banach-Stone theorem for Hilbert space-valued $L^\infty(\mu,V)$: let μ be a σ-finite measure and V a separable Hilbert space, then each isometry T of $L^\infty(\mu,V)$ onto itself has the form

$$Tx\,(s) = U(s)(\Phi x)(s)\ ,$$

where Φ extends a suitable Boolean isomorphism of μ's measure algebra and the $U(s)$ are isometries of V onto itself. Although Cambern used Hilbert space methods, it turned out that his result holds for the rather large class of all separable spaces with trivial centralizers [5]. (For the notion and properties of the centralizer $Z(X)$ of a Banach space X we refer the reader to [1].) As in the problem mentioned in the beginning, the separability of V was essential for the proof. In this article we give a generalization of Cambern's theorem into the other direction, namely, concerning the density character of V. We shall prove a Banach-Stone theorem for all Hilbert spaces, with arbitrary dimension. In fact we show more:

Theorem 2: *Let* $(\Omega_i,\Sigma_i,\mu_i)$ *be* σ*-finite non-zero measure spaces and* $V_i \neq \{0\}$ *Banach duals with trivial centralizers* ($i=1,2$)*. Then each surjective linear isometry* $T: L^\infty(\mu_1,V_1) \longleftrightarrow L^\infty(\mu_2,V_2)$ *has the form*

$$Tx\,(s) = U(s)(\Phi x)(s)\ ,$$

where Φ *extends a Boolean isomorphism of the measure algebra* Σ_1/μ_1 *onto* Σ_2/μ_2 *and* U *is a strongly measurable operator-valued function such that all* U(s) *are norm one operators from* V_1 *into* V_2.

As in [5], we shall derive Theorem 2 from a description of $Z(L^\infty(\mu,V))$ (see Theorem 1 below). We have not been able to show that the U(s) can be chosen to be surjective isometries .

A second goal of this article is the following. Apart from separability, the Banach-Stone theorem in [5] requires a trivial centralizer of V, which in particular rules out all non-trivial CK-spaces V (K compact), since $Z(CK) \simeq CK$. In the situation of vector-valued continuous function spaces $C(L,V)$ this seems to be an adequate restriction (see [1, Theorem 11.16(ii)]). In general, CK-spaces do not even have the Banach-Stone property. (We say that V has the Banach-Stone property if for each pair of compact spaces L_i the spaces $C(L_i,V)$ are isometrically isomorphic if and only if the L_i are homeomorphic.) However, for measurable function spaces we can show the following.

Theorem 4: *Let* $(\Omega_i,\Sigma_i,\mu_i)$ *be as above and* $K \neq \emptyset$ *connected and compact. Then the spaces* $L^\infty(\mu_i,CK)$ *are isometrically isomorphic if and only if the measure algebras* Σ_i/μ_i *are isomorphic.*

Although we require connectedness, this is still better than what we get in the context of vector-valued continuous function spaces. For example, C[0,1] does not have the Banach-Stone property [1, p. 143].

We mention some notations. [X] denotes the Banach space of all bounded linear operators of a Banach space X into itself. The constant function with value v is denoted by $\underline{v}$, and the characteristic function of a subset A by χ_A (where the domain of the functions is understood). If x and h are V- and [V]-valued functions resp. with the same domain, then $|x|$ and $\langle x,h\rangle$ denote the functions $t \longmapsto \|x(t)\|$ and $h(t)x(t)$, resp.. Strong measurability of h means that for all v in V the function $\langle \underline{v},h\rangle$ is measurable. Sometimes we distinguish between functions x on Ω and their equivalence classes modulo equality almost everywhere, [x]. The definition of $L^\infty(\mu,V)$ and the elementary properties that we need can be found in [3]. Since the completion of a measure does not affect the notion of (Bochner) measurability, we assume throughout that all measures are complete.

2. DUAL SPACES

The main tool in this section is a vector-valued lifting. Let $M^\infty(\mu,V)$ denote the Banach space of all bounded Bochner-measurable V-valued functions, endowed with the supremum norm $\| \ \|_\infty$. If instead we supply $M^\infty(\mu,V)$ with the essential supremum $\| \ \|_{ess}$ as seminorm, the corresponding normed space is $L^\infty(\mu,V)$. A linear $\| \ \|_{ess}$-$\| \ \|_\infty$-isometry $\sigma: L^\infty(\mu,V) \longrightarrow M^\infty(\mu,V)$ is called a *lifting*, if for each equivalence class x in $L^\infty(\mu,V)$ σx is an element of x .

Proposition 1: *Let* V *be a Banach dual. Then there is a multiplicative lifting* $\rho: L^\infty(\mu,\mathbb{K}) \longrightarrow M^\infty(\mu,\mathbb{K})$ *satisfying* $\rho\underline{1} = \underline{1}$. *For each such* ρ *there is a lifting* $\sigma: L^\infty(\mu,V) \longrightarrow M^\infty(\mu,V)$ *such that*

(1) $\sigma\underline{v} = \underline{v}$ *for all* v *in* V *and*

(2) $|\sigma x| \le \rho|x|$ *for all* x *in* $L^\infty(\mu,V)$.

Note that for arbitrary Banach spaces V it is easy to find a lifting with respect to $\| \ \|_{ess}$ on $M^\infty(\mu,V)$ (use a Hamel basis of $L^\infty(\mu,V)$). The point is that we require $\|\sigma x\|_{ess} = \|\sigma x\|_\infty$ for all x , which is not possible in general. The author is grateful to D. Fremlin for pointing out to him that c_o may serve as a counterexample.

The proof of the above proposition can be found in [6,Theorem IV.3, Propositions VI.1 and VI.2], when the scalars are real. The fact that σ selects all constant functions from their equivalence classes is not explicitly stated but immediate from the construction. Similarly, the inequality (2) is a consequence of

$$|\langle\sigma x,\underline{z}\rangle| = |\rho\langle x,\underline{z}\rangle| = \rho|\langle x,\underline{z}\rangle| \le \rho|x|$$

(z in the predual of V , $\|z\| \le 1$; see [6, p. 76 (3), p. 35 (2'), and p. 34 (IV)]. In the complex case it is easy to see that the same proof works if we replace ρ by $\tilde{\rho}(f + ig) := \rho f + i\rho g$ and observe that the multiplicativity of $\tilde{\rho}$, inherited from ρ , implies $\rho|h| = |\tilde{\rho}h|$. □

The first step in order to determine $Z(L^\infty(\mu,V))$ is the following lemma.

Lemma 1: *For* h *in* $L^\infty(\mu,[V])$ *and* x *in* $L^\infty(\mu,V)$ *define*

$$M_h x := \langle x,h\rangle$$

Then $h \longmapsto M_h$ *is an isometric embedding of* $L^\infty(\mu,[V])$ *into* $[L^\infty(\mu,V)]$, *mapping* $L^\infty(\mu,Z(V))$ *into* $Z(L^\infty(\mu,V))$.

Proof. Obviously M_h is well-defined and satisfies $\|M_h\| \le$

$\|h\|_{ess}$. For the reverse inequality it suffices to show that the (linear) mapping $h \longmapsto M_h$ is isometric on the dense subspace of countably valued functions. This however is clear - for $h = \sum_{i=1}^{\infty} R_i \chi_{A_i}$ look at $x := \sum_{i=1}^{\infty} v_i \chi_{A_i}$ with $\|v_i\| = 1$, $\|R_i v_i\| \geq \|R_i\| - \varepsilon$ (w.l.o.g. $V \neq \{0\}$). The proof of the inclusion $L^{\infty}(\mu, Z(V)) \subset Z(L^{\infty}(\mu, V))$ is essentially contained in [5, Proposition 1] (replace "strongly measurable" by "measurable"). □

Theorem 1: *Let* V *be a dual space. Then* $L^{\infty}(\mu, Z(V)) \simeq Z(L^{\infty}(\mu, V))$ *under the embedding of Lemma 1.*

The proof is a simplified version of [5, Theorem 1]. We have to show "$\supset$". First we restrict ourselves to the case $\mathbb{K} = \mathbb{R}$. Namely, if for $\mathbb{K} = \mathbb{C}$ we denote by $X_{\mathbb{R}}$ the underlying real space of a Banach space X , then $L^{\infty}(\mu, Z(V)) = L^{\infty}(\mu, Z(V_{\mathbb{R}})) + i\, L^{\infty}(\mu, Z(V_{\mathbb{R}}))$ and $Z(L^{\infty}(\mu, V)) = Z(L^{\infty}(\mu, V_{\mathbb{R}})) + i\, Z(L^{\infty}(\mu, V_{\mathbb{R}}))$ [1, Theorem 3.13(i)]. For the rest of this proof we distinguish between measurable functions $x : \Omega \longrightarrow V$ and their equivalence classes $[x]$. Let $R \in Z(L^{\infty}(\mu, V))$, w.l.o.g. $\|R\| = 1$. Choose a lifting σ as in Proposition 1 and define an operator R_t on V $(t \in \Omega)$ by

$$R_t v := \sigma(R[\underline{v}])(t)$$

Evidently R_t is linear, $\|R_t\| \leq 1$, and the mapping $t \longmapsto R_t$ is strongly measurable. In order to verify $R_t \in Z(V)$ it suffices to show that

$$\|u \pm v\| \leq \alpha \quad \text{implies} \quad \|u \pm R_t v\| \leq \alpha \quad (u, v \in V, \quad \alpha > 0)$$

[1, Theorem 3.12]. Now $\|[\underline{u}] \pm [\underline{v}]\| = \|u \pm v\| \leq \alpha$ implies $\|[\underline{u}] \pm R[\underline{v}]\| \leq \alpha$ [1, loc. cit.], hence

$\|u \pm R_t v\| = \|\rho[\underline{u}](t) \pm \rho(R[\underline{v}])(t)\| \leq \|\rho([\underline{u}] \pm R[\underline{v}])\| \leq \alpha$.

Thus $t \longmapsto h(t) := R_t$ is a strongly measurable bounded mapping with values in $Z(V)$. Since V is a dual, the norm and strong topologies on $Z(V)$ coincide [1, p.155, Example 5]. Lemma 3 in [5] then shows that h is Bochner measurable, hence an element of $L^{\infty}(\mu, Z(V))$. It remains to show $M_h = R$. M_h and R coincide on the constant functions. Since both operators commute with the characteristic projections $x \longmapsto \chi_A x$, $A \in \Sigma$, they coincide on all countably valued functions, hence everywhere in $L^{\infty}(\mu, V)$. □

Now we shall prove Theorem 2. Since the centralizers of V_i are trivial, i.e. $Z(V_i) \simeq \mathbb{K}$, the conclusion of Theorem 1 is $Z(L^{\infty}(\mu_i, V_i)) \simeq L^{\infty}(\mu_i)$. Thus the isometry $T : L^{\infty}(\mu_1, V_1) \longleftrightarrow L^{\infty}(\mu_2, V_2)$ induces an isometry between $L^{\infty}(\mu_1)$ and $L^{\infty}(\mu_2)$ that can be extended to an isometry Φ of $L^\infty(\mu_1,V_1)$ onto $L^\infty(\mu_2,V_1)$ in such a way that the isometry $S := T\circ\Phi^{-1}: L^\infty(\mu_2,V_1) \longleftrightarrow L^\infty(\mu_2,V_2)$ satisfies

(3) $$S\chi_A y = \chi_A Sy \qquad (y \in L^\infty(\mu_2,V_2),\ A \in \Sigma_2)$$

(see [5] for details).

It remains to show that S has the form

(4) $$Sy(s) = U(s)y(s)$$

with U as in the statement of the theorem. Let ρ and σ_i be liftings of $L^\infty(\mu_2)$ and $L^\infty(\mu_2,V_i)$ resp. as in Proposition 1 $(i=1,2)$. (3) implies that $|Sy| = |y|$ a.e.. This together with (2) gives

(5) $$|\sigma_2(Sy)| \le \rho|Sy| = \rho|y|$$

Now define $$U(s)v := \sigma_2(S\underline{v})(s) \quad .$$

Trivially $U(s)$ is linear and U is strongly measurable. From (5) it follows that $\|U(s)v\| \le \|v\|$, and, since $|U(\cdot)v| = \|v\|$ a.e. for any $v \neq 0$, we have $\|U(s)\| = 1$ for all s outside a null set N. For $s \in N$ replace $U(s)$ by any norm one operator from V_1 into V_2. In order to verify (4) we note that the strong measurability of U implies that also $U(\cdot)y(\cdot)$ is measurable, and evidently $\tilde{S}y(s) := U(s)y(s)$ defines a bounded operator $\tilde{S}$ of $L^\infty(\mu_2,V_1)$ into $L^\infty(\mu_2,V_2)$ that coincides with S on all countably valued functions, hence $S = \tilde{S}$. □

3. CK-SPACES

In order to argue as in the proof of Theorem 2 we have to replace $Z(L^\infty(\mu,V))$ by a subspace isomorphic to $L^\infty(\mu)$. The *Cunningham ∞-algebra* $C_\infty(X)$ of a Banach space X is the closed subspace of $Z(X)$ generated by the idempotents of $Z(X)$. These idempotents are exactly the *M-projections*, i.e. projections P satisfying $\|x\| = \max\{\|Px\|, \|x-Px\|\}$ $(x \in X)$ [1, pp. 31 and 72].

Proposition 2: *Assume the conclusion of Theorem 1 holds. Then the M-projections of $L^\infty(\mu,V)$ are exactly those elements of $L^\infty(\mu,Z(V))$ whose values are M-projections of V almost everywhere.*

Proof. $L^\infty(\mu,Z(V))$ is a Banach algebra with the pointwise multiplication, and the mapping $M_\cdot$ is obviously multiplicative. Since the M-projections are the idempotents, the statement of the proposition is just the trivial fact that $h^2 = h$ if and only if $h(t)^2 = h(t)$ a.e.. □

Theorem 3: *Let K be compact. Then under the embedding of Lemma 1*

$$L^\infty(\mu,CK) \simeq Z(L^\infty(\mu,CK))$$

$$L^\infty(\mu) \simeq C_\infty(L^\infty(\mu,CK)) \qquad \textit{if } K \textit{ is connected.}$$

Proof. As an abstract M-space with unit, $L^\infty(\mu,CK)$ is isometrically isomorphic to its centralizer. However, we can see more directly that the embedding $M_\bullet$ maps $L^\infty(\mu,Z(CK)) = L^\infty(\mu,CK)$ onto $Z(L^\infty(\mu,CK))$, if for R in the latter space we look at $h := R(\underline{1})$, where $\underline{1}$ is the constant function on Ω taking the constant function $\underline{1}$ on K as value: Since for all $g \in L^\infty(\mu,CK)$ M_g is in the centralizer, it commutes with R, and so we have

$$Rg = R(M_g(\underline{1})) = M_g(R(\underline{1})) = \langle h,g\rangle = M_h g\ ,$$

hence $R = M_h$. (Observe that the action of $g(t) \in CK$ as an element of $Z(CK)$ is just the multiplication in CK.)
As to *b)*, the above proposition shows that $M_\bullet$ maps $C_\infty(L^\infty(\mu,CK))$, the space generated by the M-projections, into $L^\infty(\mu,C_\infty(CK))$, which is isomorphic to $L^\infty(\mu)$, since CK has only trivial idempotents. Since $L^\infty(\mu)$ is generated by the simple functions and these correspond to finite linear combinations of characteristic projections in $L^\infty(\mu,CK)$, which are clearly M-projections, the reverse inclusion is also shown. □

Now we can easily prove Theorem 4. The "if" part is straightforward (see [5]). Conversely, if $T:L^\infty(\mu_1,CK) \longleftrightarrow L^\infty(\mu_2,CK)$ is an isometry, the corresponding isometry between the operator spaces, $\Phi R := T \circ R \circ T^{-1}$, sends M-projections into M-projections and consequently maps $C_\infty(L^\infty(\mu_1,CK)) \simeq L^\infty(\mu_1)$ onto $C_\infty(L^\infty(\mu_2,CK)) \simeq L^\infty(\mu_2)$. The classical Banach-Stone theorem for $L^\infty(\mu)$ then says that the Boolean algebras Σ_i/μ_i are isomorphic. More directly, if we restrict Φ to the Boolean algebra of all M-projections of $L^\infty(\mu_1,CK)$ which in view of Proposition 2 is isomorphic to Σ_1/μ_1, we have the desired isomorphism. □

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