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TENSOR PRODUCTS OF LINEAR OPERATORS IN LOCALLY CONVEX SPACES

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Given continuous linear operators $T_i : E_i \longrightarrow E_i$ on locally convex spaces E_i ($i = 1,2$) and a polynomial P in two variables, spectral properties of polynomial operators

$$P(T_1 \hat{\otimes} I_2, I_1 \hat{\otimes} T_2) : E_1 \hat{\otimes} E_2 \longrightarrow E_1 \hat{\otimes} E_2$$

are studied in dependence of the spectral properties of their components T_1 and T_2. Here $E_1 \hat{\otimes} E_2$ denotes the completion of the tensor product $E_1 \otimes E_2$ with respect to a suitable tensor product topology lying between the ε- and the π-topology, and I_i denotes the identity map on E_i.

One of the main problems is to establish spectral mapping theorems of the form

$$\text{(i)} \quad P(\sigma(T_1;\ldots), \sigma(T_2;\ldots)) \subset \sigma(P(T_1 \hat{\otimes} I_2, I_1 \hat{\otimes} T_2);\ldots)$$

and

$$\text{(ii)} \quad P(\sigma(T_1;\ldots), \sigma(T_2;\ldots)) \supset \sigma(P(T_1 \hat{\otimes} I_2, I_1 \hat{\otimes} T_2);\ldots)$$

where $\sigma(S;\ldots) := \{\lambda \in \mathbb{C} : \lambda - S$ has no inverse in $\ldots\}$ for $S \in L(F)$ denote suitable spectra depending on subsets ... from the algebra $L(F)$.

In [3] A. Brown and C. Pearcy established (i) and (ii) for $P(z_1,z_2) = z_1 z_2$ in the Hilbert space setting, and M. Schechter [23] proved both for bounded linear operators on Banach spaces and general polynomials P. The case of unbounded, closed operators on Banach spaces, which arises from problems in evolution equations (cf. [2]), differential equations with operator coefficients (cf. [4]), and N-body problems in quantum mechanics, has been investigated by T. Ichinose [13] - [17] and M. Reed and B. Simon [21]. It turns out that (i) is always true, whereas (ii) in general fails even if the left hand side of (ii) is replaced by its closure in $\mathbb{C}$.

Since many problems for unbounded, closed operators on Banach spaces admit a reformulation in a locally convex setting with continuous linear operators, this may draw some attention to the situation studied in this paper, too.

This article is based on a part of the author's Habilitationsschrift [28].

0. Preliminaries. We start with some basic algebraic notions. Let A denote an algebra with unit element e over the complex numbers $\mathbb{C}$, and let M be a subset of A. For $a_i \in A$ $(i = 1,2,\dots,n)$ denote by $\rho(a_1,a_2,\dots,a_n;M)$ the set of all those $(\lambda_1,\lambda_2,\dots,\lambda_n) \in \mathbb{C}^n$ such that there exist $b_i \in M$ $(i = 1,2,\dots,n)$ with

$$e = b_1(\lambda_1 e - a_1) + b_2(\lambda_2 e - a_2) + \dots + b_n(\lambda_n e - a_n).$$

The set

$$(0.1)\quad \sigma(a_1,a_2,\dots,a_n;M) := \mathbb{C}^n \setminus \rho(a_1,a_2,\dots,a_n;M)$$

is called *joint spectrum* of $(a_1,a_2,\dots,a_n)$ with respect to M. If M = A this notion is well known from Banach algebra theory, but it turns out that for purposes of locally convex algebra theory one has to choose smaller sets M (cf. [28]). Throughout this paper we will restrict our attention to commutative subalgebras of the locally convex algebra $L_s(E)$ of all continuous linear operators on a locally convex space E equipped with the topology of pointwise convergence.

If Γ denotes a fundamental system of continuous semi-norms E, let

$$(0.2)\quad G(E;\Gamma) := \{T \in L(E) : \exists\ c > 0 \text{ s.t. } p \circ T \leq cp \text{ for all } p \in \Gamma\}.$$

If E is Mackey-complete, then $G(E,\Gamma)$ is a Banach algebra, when equipped with the norm

$$(0.3)\quad \|T\|_\Gamma := \sup\{\sup\{p(Tx) : x \in E,\ p(x) = 1\} : p \in \Gamma\} \text{ (cf. [19]).}$$

Lateron, we shall deal with a decomposition of $\sigma(T;G(E;\Gamma))$ for $T \in L(E)$. For that purpose let

$$\Lambda(T;\Gamma) := \{\lambda \in \mathbb{C} : \exists\ c > 0 \text{ s.t. } p \circ (T - \lambda) \geq cp \text{ for all } p \in \Gamma\}$$

and

$$(0.4)\quad \pi(T;\Gamma) := \mathbb{C} \setminus \Lambda(T;\ )$$

$$(0.5)\quad \gamma(T;\Gamma) := \{\zeta \in \Lambda(T;\Gamma) : \overline{(T - \zeta)E}^E \neq E\}.$$

Moreover let

$$\Lambda(T) := \bigcup_\Gamma \Lambda(T;\Gamma)$$

where the union runs over all fundamental systems of continuous semi-norms on E, and let

$$(0.6)\quad \pi(T) := \mathbb{C} \setminus \Lambda(T)$$

(0.7) $\quad \gamma(T) := \{\zeta \in \Lambda(T) : \overline{(T-\zeta)E}^{E} \neq E\}$

0.1. Remarks.

(1) The sets $\pi(T;\Gamma)$, $\pi(T)$ and $\gamma(T;\Gamma)$, $\gamma(T)$ are very similar to Halmos' notions of *approximate point spectrum* and *compression spectrum* (cf. [13], [18]).

(2) For every $T \in L(E)$ we have $\rho(T;G(E;\Gamma)) \subset \Lambda(T;\Gamma)$ and hence $\rho(T;G(E)) \subset \Lambda(T)$, where

(0.8) $\quad G(E) := \bigcup_{\Gamma} G(E;\Gamma)$,

the union taken over all fundamental systems of continuous semi-norms on E.

(3) Let $T \in L(E)$, and let $\tilde{T}$ denote the (unique) continuous extension of T onto the completion $\tilde{E}$ of E. Then

$$\pi(\tilde{T};\Gamma) = \pi(T;\Gamma), \quad \pi(\tilde{T}) = \pi(T)$$
$$\gamma(\tilde{T};\Gamma) = \gamma(T;\Gamma), \quad \gamma(\tilde{T}) = \gamma(T) \quad \text{(cf. [28])}.$$

(4) If E is a complete locally convex space, and $T \in L(E)$, then

$$\sigma(T;G(E;\Gamma)) = \pi(T;\Gamma) \cup \gamma(T;\Gamma)$$
$$\sigma(T;G(E)) = \pi(T) \cup \gamma(T) \quad \text{(cf. [28])}.$$

If A is a subset of L(E), let

(0.9) $\quad A_b := A \cap G(E)$.

The elements of A_b will be called *Allan-bounded operators* in order to distinguish them from bounded operators (cf. [1], [19], [29]).

The following lemma can be proved by means of Gelfand theory (see [28] for a more general result).

0.2. LEMMA. *Let E denote a Mackey-complete locally convex space, and let $A \subset L(E)$ be a commutative subalgebra containing I_E. Let $T_1, T_2 \in A$, let P be a polynomial in two variables, and suppose that neither $\sigma(T_1;G(E;\Gamma) \cap A)$ nor $\sigma(T_2;G(E;\Gamma) \cap A)$ covers the whole plane $\mathbb{C}$. Then*

$$P(\sigma(T_1,T_2;A \cap G(E;\Gamma))) \subset \sigma(P(T_1,T_2);A \cap G(E;\Gamma)).$$

In general this inclusion is strict, but there is equality, if T_1, $T_2 \in G(E;\Gamma)$.

1. Tensor products and admissible topologies. Let E_1 and E_2 be locally convex spaces, and let $E_1 \otimes E_2$ denote their algebraic tensor product. A semi-norm p on $E_1 \otimes E_2$ is called *cross-semi-norm*, provided there exist continuous semi-norms p_i on E_i (i = 1,2) such that

(1.1) $\quad p(x_1 \otimes x_2) = p_1(x_1)p_2(x_2)$ for every $x_1 \otimes x_2 \in E_1 \otimes E_2$.

If (1.1) holds true, we shall abbreviate this by writing $p = p_1 \otimes p_2$. By $E_1 \otimes_\alpha E_2$ we denote $E_1 \otimes E_2$ equipped with a locally convex topology α. If Γ is a fundamental system of continuous semi-norms on $E_1 \otimes_\alpha E_2$ we call Γ a *D-fundamental system*, provided Γ consists of cross-semi-norms only, and if there exists a constant $k(\Gamma) > 0$ such that

(1.2) $\quad p = p_1 \otimes p_2$ implies $p \geq k(\Gamma) p_1 \otimes_\varepsilon p_2$ for all $p \in \Gamma$,

where $p_1 \otimes_\varepsilon p_2$ denotes a canonical semi-norm of the ε-tensor product (cf. [22]). More specially we shall consider only those locally convex topologies α on $E_1 \otimes E_2$ fulfilling the following conditions:

(i) There exists a D-fundamental system on $E_1 \otimes_\alpha E_2$.

(1.3) *(ii) If $A_i \subset L(E_i)$ $(i = 1,2)$ are equicontinuous subsets, then $A_1 \otimes A_2 \subset L(E_1 \otimes_\alpha E_2)$ is equicontinuous.*

1.1. Remarks.

(1) The letter "D" in D-fundamental system is to suggest "dualizable" since in the normed case (1.2) with $k(\Gamma) = 1$ implies, that the dual norm is a cross-norm, too.

(2) If Γ is a D-fundamentalsystem, and if $p_1 \otimes p_2 = p \in \Gamma$, then

(1.4) $\quad U_{p_1}{}^{o} \otimes U_{p_2}{}^{o} \subset k(\Gamma)^{-1} U_p{}^{o}$.

(3) For $\alpha = \varepsilon$, $\alpha = \pi$ or more general for locally convex tensornorm-topologies as studied by Harksen [10], condition (1.2) is automatically fulfilled with constant $k(\Gamma) = 1$.

(4) For every cross-semi-norm $p = p_1 \otimes p_2$ one has $p \leq p_1 \otimes_\pi p_2$.

(5) Condition (1.3) (ii) especially implies $E_1' \otimes E_2' \subset (E_1 \otimes_\alpha E_2)'$.

By $T_1 \hat{\otimes} T_2$ we denote the extension of $T_1 \otimes T_2$ onto the completion $E_1 \hat{\otimes}_\alpha E_2$ of $E_1 \otimes_\alpha E_2$.

In order to avoid technical difficulties, we make the following general assumption :

E_1, E_2 and $E_1 \hat{\otimes}_\alpha E_2$ are barrelled locally convex spaces, and E_1, E_2 are complete.

As we have announced in the introduction, we have to establish relations between spectra of operators on tensor products and spectra of their components. In order to do so, we start with a simple

1.2. LEMMA. *Let $T_1 \in L(E_1)$, and let α denote a tensor product topology fulfilling* (1.3) *on $E_1 \otimes E_2$. Then we have*

(i) $\sigma(T_1 \hat{\otimes} I_2; L(E_1 \hat{\otimes}_\alpha E_2)) = \sigma(T_1; L(E_1))$

(ii) $\sigma(T_1 \hat{\otimes} I_2; (L_s(E_1 \hat{\otimes}_\alpha E_2))_b) = \sigma(T_1; (L_s(E_1))_b)$,

and consequently the sets $\sigma(T_1 \hat{\otimes} I_2; L(E_1 \hat{\otimes}_\alpha E_2))$ *and*

$\sigma(T_1 \hat{\otimes} I_2;(L_s(E_1 \hat{\otimes}_\alpha E_2))_b)$ *are independent of the special topology* α.

One of the main difficulties when dealing with operators on tensor products of locally convex spaces originates from the fact, that not every Allan-bounded operator $C \in L_s(E_1 \hat{\otimes}_\alpha E_2)$ is already contained in some $G(E_1 \hat{\otimes}_\alpha E_2;\Gamma)$, where Γ is a D-fundamental system (cf. [28], p. 35 for an example). Therefore we consider the following subclasses of operators:

1.3. DEFINITION. For a subset $A \subset L_s(E_1 \hat{\otimes}_\alpha E_2)$ let

$$A_{cb} := \bigcup_\Gamma (A \cap G(E_1 \hat{\otimes}_\alpha E_2;\Gamma)),$$

where the union is taken over all D-fundamental system Γ on $E_1 \hat{\otimes}_\alpha E_2$ The elements of A_{cb} are called *Cross-semi-norm bounded operators*. Moreover, an operator $C \in L(E_1 \hat{\otimes}_\alpha E_2)$ is said to be *cross-semi-norm stable*, provided the following condition is fulfilled:
For every $R \in (\{C;L(E_1 \hat{\otimes}_\alpha E_2)\}^{cc})_{cb}$ ($\{\dots\}^{cc}$ *denoting the bicommutant of* $\{\dots\}$ *in* $L(E_1 \hat{\otimes}_\alpha E_2)$) *there exists a D-fundamental system* Γ *such that* $C, R \in G(E_1 \hat{\otimes}_\alpha E_2;\Gamma)$.
Let

$$A_{cs} := \{C \in A : C \text{ is cross-semi-norm stable}\}.$$

For every subset $A \subset L_s(E_1 \hat{\otimes}_\alpha E_2)$ we have $A_{cs} \subset A_{cb} \subset A_b$. If A is a commutative algebra, then so is A_b, but we do not know whether A_{cs} or A_{cb} are. For our purposes the following result will be sufficient

1.4. LEMMA. *Let* $A \subset L_s(E_1 \hat{\otimes}_\alpha E_2)$ *denote a commutative subalgebra. Then the following hold true:*

(i) $A_{cs} + A_{cs} \subset A_{cb}$

(ii) $A_{cs} \circ A_{cs} \subset A_{cb}$

(iii) $\{\sum_{finite} T_i \hat{\otimes} S_i : T_i \in (L_s(E_1))_b, S_i \in (L_s(E_2))_b, T_i \hat{\otimes} S_i \in A\} \subset A_{cs}$

Proof. Since (i) and (ii) follow immediately from the definitions, only (iii) needs a proof. So let $T_i \hat{\otimes} S_i \in A$, T_i and S_i Allan-bounded ($i = 1,2,\dots,k$), let Γ denote any D-fundamental system, let $p \in \Gamma$ with $p = p_1 \otimes p_2$, and let $R \in A \cap G(E_1 \hat{\otimes}_\alpha E_2;\Gamma)$. Let

$$\hat{p}(z) := \sup\{p(T_1^{n_1} \hat{\otimes} S_1^{n_2} z) : n_1, n_2 \geq 0\}.$$

Then $\hat{p}$ is a continuous cross-semi-norm, for we have

$$\hat{p}(x_1 \otimes x_2) = \sup\{p_1(T_1^{n_1} x_1) p_2(S_1^{n_2} x_2) : n_1, n_2 \geq 0\}$$
$$= \hat{p}_1(x_1)\hat{p}_2(x_2).$$

On the other hand

$$\hat{p}(Rz) = \sup\{p(R(T_1^{n_1} \hat{\otimes} S_1^{n_2})z) : n_1, n_2 \geq 0\} \leq \|R\|_\Gamma \hat{p}(z),$$

and therefore $R \in G(E_1 \hat{\otimes}_\alpha E_2; \hat{\Gamma})$. But Proposition 1.5 below tells us that $\hat{\Gamma}$ is a D-fundamental system, and so we are done by repeating the above argument k-times.//

1.5. PROPOSITION. *Let $G_i \subset L(E_i)$ $(i = 1,2)$ denote equicontinuous subsets such that $G_i \circ G_i \subset G_i$. Then for every D-fundamental system Γ_o on $E_1 \hat{\otimes}_\alpha E_2$ there exists another D-fundamental system Γ, such that*

$$\sup\{\|T \hat{\otimes} S\|_\Gamma : T \in G_1, S \in G_2\} \leq 1.$$

Proof. Let $p \in \Gamma_o$ such that

$$(*) \qquad p = p_1 \otimes p_2 \text{ and } p \geq k(\Gamma_o) p_1 \hat{\otimes}_\varepsilon p_2.$$

Then $\hat{p}_i(x_i) := \sup\{p_i(T_i x) : T_i \in G_i\}$ define continuous semi-norms on E_i such that $\hat{p}_i \geq p_i$ and $\hat{p}_i \circ R \leq \hat{p}_i$ for all $R \in G_i$ $(i = 1,2)$. Next let

$$\hat{p}(z) := \sup\{p(T \hat{\otimes} Iz) : T \in G_1\}$$

$$\widehat{(p_1 \hat{\otimes}_\varepsilon p_2)}(z) := \sup\{(p_1 \hat{\otimes}_\varepsilon p_2)(T \hat{\otimes} Iz) : T \in G_1\}.$$

Because of (*) we obtain

$$\hat{p} \geq k(\Gamma_o) \widehat{p_1 \hat{\otimes}_\varepsilon p_2} \text{ and } \hat{p} = \hat{p}_1 \otimes p_2.$$

On the other hand

$$\hat{p}_1(x) = \sup\{|(\phi, Tx)| : T \in G_1, \phi \in U_{p_1}^{o}\}$$
$$= \sup\{|(\psi, x)| : \psi \in U_{p_1}^{o}\}$$

and hence

$$\widehat{p_1 \hat{\otimes}_\varepsilon p_2} = \hat{p}_1 \otimes_\varepsilon p_2, \ \hat{p} = \hat{p}_1 \otimes p_2, \text{ and } \hat{p} \geq k(\Gamma_o) \hat{p}_1 \hat{\otimes}_\varepsilon p_2.$$

By repeating this argument for G_2 and p_2, we obtain the desired result.//

2. Joint spectra of tensor products of operators. Our main result in this section will be the following

2.1. THEOREM. *Let E_1 and E_2 denote two locally convex spaces, let α be a tensor product topology fulfilling (1.3), let $T_i \in L(E_i)$ $(i = 1,2)$, and let A denote the bicommutant of $T_1 \hat{\otimes} I_2$, $I_1 \hat{\otimes} T_2$ in the algebra $L_s(E_1 \hat{\otimes}_\alpha E_2)$. Then*

$$\sigma(T_1 \hat{\otimes} I_2, I_1 \hat{\otimes} T_2; A_{cs}) = \sigma(T_1; (L_s(E_1))_b) \times \sigma(T_2; (L_s(E_2))_b)$$

2.2. Remark. If E_1 and E_2 are Banach spaces, then 2.1 generalizes slightly a result of Dash and Schechter [6] to quasi-uniform cross-norms in the sense of Ichinose [17]. In this setting it turns out, that 2.1 holds also true, if A denotes the (bigger) commutant of $T_1 \hat{\otimes} I_2$, $I_1 \hat{\otimes} T_2$ in $L_s(E_1 \hat{\otimes}_\alpha E_2)$. Indeed this sharpened version of 2.1 can be proved for bounded operators on Fréchet spaces E_1 and E_2. For this and related results concerning the "classical" bicommutant spectra, the reader is referred to [28], II.3.

In order to prove 2.1, we need the following

2.3. LEMMA. *Let* $T_i \in L(E_i)$ $(i = 1,2)$, *let* $(\lambda,\gamma) \in \pi(T_1) \times \pi(T_2)$ *(cf. (0.6)), and let* Γ *denote a D-fundamental system on* $E_1 \hat{\otimes}_\alpha E_2$. *Then for* $\varepsilon > 0$ *given, the set*

$$\Gamma_\varepsilon := \{p \in \Gamma : \exists\ x \otimes y \in E_1 \otimes E_2 \ s.t.\ \varepsilon p(x \otimes y) > \max\{p((T_1-\lambda)x \otimes y), p(x \otimes (T_2-\gamma)y)\}\}$$

is also a D-fundamental system.

Proof. Let $q \in \Gamma$ be given. If for every $p \in \Gamma$ such that $p = p_1 \otimes p_2$ and $p \geq q$, we had $\varepsilon p_1(x) p_2(y) < p_1((T_1 - \lambda)x) p_2(y)$ for all $x \otimes y$, then $\lambda \in \mathcal{A}(T_1)$ contradicting our assumptions. Therefore

$$\tilde{\Gamma}_\varepsilon := \{p \in \Gamma : \exists\ x \otimes y \in E_1 \otimes E_2 \ s.t.\ \varepsilon p(x \otimes y) > p(T_1 - \lambda)x \otimes y)\}$$

is a D-fundamental system. But for every $p \in \tilde{\Gamma}_\varepsilon$ such that $p = p_1 \otimes p_2$ there exists $x_1 \in E_1$ such that

$$\varepsilon p(x_1 \otimes y) > p((T_1 - \lambda)x_1 \otimes y) \text{ for all } y \in E_2 \text{ such that } p_2(y) \neq 0.$$

By repeating the argument above for $\tilde{\Gamma}_\varepsilon$ replacing Γ and $T_2 - \gamma I_2$, we obtain the desired result. //

Proof of 2.1.

1° By 1.4 (iii) the operators $(T_1 - \lambda)^{-1} \hat{\otimes} I_2$ and $I_1 \hat{\otimes} (T_2 - \gamma)^{-1}$ belong to A_{cs}, provided they are Allan-bounded only. Therefore

$$\sigma(T_1 \hat{\otimes} I_2, I_1 \hat{\otimes} T_2; A_{cs}) \subset \sigma(T_1 \hat{\otimes} I_2; A_b) \times \sigma(I_1 \hat{\otimes} T_2; A_b).$$

By 1.2 (ii) this gives one half of the Theorem. In order to prove the inclusion "$\supset$" let

2° $(\lambda,\gamma) \in \pi(T_1) \times \pi(T_2)$. Suppose $(\lambda,\gamma) \in \rho(T_1 \hat{\otimes} I_2, I_1 \hat{\otimes} T_2; A_{cs})$. Then there exist $C_1, C_2 \in A_{cs}$ such that

(2.1) $I_1 \hat{\otimes} I_2 = C_1((T_1 - \lambda) \hat{\otimes} I_2) + C_2(I_1 \hat{\otimes} (T_2 - \gamma))$.

It follows from the definition of A_{cs} that there exist a D-fundamental system Γ and constants c_1, c_2 such that

(2.2) $p \circ C_1 \leq c_1 p$ and $p \circ C_2 \leq c_2 p$ for all $p \in \Gamma$.

Choose $0 < \varepsilon < (4(c_1 + c_2))^{-1}$ and apply Lemma 2.3. Hence there exists a D-fundamental system $\Gamma_\varepsilon \subset \Gamma$ such that for every $p \in \Gamma_\varepsilon$ we find $\bar{x} \otimes \bar{y}$ obeying

$$\varepsilon p(\bar{x}\otimes\bar{y}) > \max\{p((T_1 - \lambda)\bar{x}\otimes\bar{y}), p(\bar{x}\otimes(T_2 - \gamma)\bar{y})\}.$$

But if we evaluate equation (2.1) in $\bar{x}\otimes\bar{y}$ and then apply p, we obtain by considering (2.2)

$$p(\bar{x}\otimes\bar{y}) \leq c_1 p((T_1 - \lambda)\bar{x}\otimes\bar{y}) + p(\bar{x}\otimes(T_2 - \gamma)\bar{y}) <$$
$$< \varepsilon c_1 p(\bar{x}\otimes\bar{y}) + \varepsilon c_2 p(\bar{x}\otimes\bar{y}) < 4^{-1}p(\bar{x}\otimes\bar{y})$$

and hence a contradiction, which gives

$$\pi(T_1)\times\pi(T_2) \subset \sigma(T_1\hat{\otimes}I_2, I_1\hat{\otimes}T_2; A_{cs}).$$

3^o $(\lambda,\gamma)\in \gamma(T_1)\times\gamma(T_2)$: Choosing

$\phi \in ((T_1 - \lambda)E_1)^o$, $x_1 \in E_1$ such that $\phi(x_1) = 1$ and

$\psi \in ((T_2 - \gamma)E_2)^o$, $x_2 \in E_2$ such that $\psi(x_2) = 1$,

we find that $\phi\hat{\otimes}\psi$ vanishes on both $(T_1 - \lambda)\hat{\otimes}I_2(E_1\hat{\otimes}_\alpha E_2)$ and $I_1\hat{\otimes}(T_2 - \gamma)(E_1\hat{\otimes}_\alpha E_2)$. Therefore (2.1) cannot be fulfilled by some pair $(C_1,C_2)\in A^2$: evaluate (2.1) in $x_1\otimes x_2$ and apply $\phi\hat{\otimes}\psi$; then the left-hand side gives 1, whereas the right-hand side vanishes. So we have

$$\gamma(T_1)\times\gamma(T_2) \subset \sigma(T_1\hat{\otimes}I_2, I_1\hat{\otimes}T_2; A).$$

4^o $(\lambda,\gamma)\in \pi(T_1)\times\gamma(T_2)$ (and symmetrically $(\lambda,\gamma)\in\gamma(T_1)\times\pi(T_2)$): Suppose $(\lambda,\gamma)\in \rho(T_1\hat{\otimes}I_2, I_1\hat{\otimes}T_2; A_{cs})$. Then (2.1) is fulfilled for suitable $C_1, C_2 \in G(E_1\hat{\otimes}_\alpha E_2; \Gamma)$, Γ denoting some D-fundamental system. Choose $0 < \varepsilon < k(\Gamma)(4\|C_1\|_\Gamma)^{-1}$. Since $(T_2 - \gamma)E_2$ is not dense in E_2, there exists a continuous semi-norm p_2 on E_2 such that $\overline{(T_2 - \gamma)E_2}$ is not U_{p_2}-dense. If $p_1 \neq 0$ is a continuous semi-norm on E_1, we find $p_o \in \Gamma_\varepsilon$ such that $p_o \geq p_1 \hat{\otimes}_\varepsilon p_2$ and a constant $c > 0$ such that $p_{o2} \geq cp_2$, where $p_o = p_{o1}\otimes p_{o2}$. Therefore $\overline{(T_2 - \gamma)E_2}$ is not p_{o2}-dense, too, and we find by considering the quotient space $E_2/\overline{(T_2 - \gamma)E_2}$ an $x_2 \in E_2$ such that $p_{o2}(x_2) = 1$ and $\psi \in U_{p_{o2}}^o$ such that $\psi(x_2) = 2^{-1}$. By 2.3 we find $x_1\otimes y_1 \in E_1\otimes E_2$ such that

$$\varepsilon p_o(x_1\otimes y_1) > p_o((T_1 - \lambda)x_1\otimes y_1).$$

Since p_o is a cross-semi-norm this implies

$$\varepsilon p_o(x_1\otimes x_2) > p_o((T_1 - \lambda)x_1\otimes x_2).$$

Now choose $\phi \in U_{p_{o1}}^o$ such that $\phi(x_1) = p_{o1}(x_1)$. Then the extension $\phi\hat{\otimes}\psi$ of $\phi\otimes\psi$ onto the completion lies in $U_{p_o}^o$ by (1.4). So evaluate (2.1) in $x_1\otimes x_2$ and apply $\phi\hat{\otimes}\psi$. Then we obtain taking into consideration $(\phi\hat{\otimes}\psi)(C_2(x_1\otimes(T_2 - \gamma)x_2) = 0$:

$$p_{o1}(x_1)2^{-1} = \phi(x_1)\psi(x_2) = (\phi\hat{\otimes}\psi)(C_1((T_1 - \lambda)x_1\otimes x_2)) \leq$$

$$\leq k(\Gamma)^{-1} p_o(C_1((T_1 - \lambda)x_1 \otimes x_2))$$
$$\leq k(\Gamma)^{-1} \| C_1 \|_\Gamma \, p_o((T_1 - \lambda)x_1 \otimes x_2)$$
$$< 4^{-1} p_o(x_1 \otimes x_2) = 4^{-1} p_{o1}(x_1)$$

and hence a contradiction. By Remark 0.1 (4) we are done.//

In doing the proof of 2.1 with a fixed D-fundamental system Γ and taking into consideration 0.1 (4), we get the following variant of 2.1

2.4. COROLLARY. *Suppose the assumptions of 2.1 are fulfilled. Then for every fixed D-fundamental system Γ on $E_1 \hat{\otimes}_\alpha E_2$, we have*

$$\sigma(T_1 \hat{\otimes} I_2, I_1 \hat{\otimes} T_2; A \cap G(E_1 \hat{\otimes}_\alpha E_2; \Gamma)) =$$
$$= \sigma(T_1 \hat{\otimes} I_2; A \cap G(E_1 \hat{\otimes}_\alpha E_2; \Gamma)) \times \sigma(I_1 \hat{\otimes} T_2; A \cap G(E_1 \hat{\otimes}_\alpha E_2; \Gamma))$$
$$= \sigma(T_1; G(E_1; \Gamma_1)) \times \sigma(T_2; G(E_2; \Gamma_2)),$$

where $\Gamma_1 := \{p(\cdot \otimes x_2) : p \in \Gamma\}$, $\Gamma_2 := \{p(x_1 \otimes \cdot) : p \in \Gamma\}$ with $0 \neq x_1 \otimes x_2 \in E_1 \otimes E_2$ fixed.

For a given D-fundamental system Γ let Γ_1, Γ_2 denote the fundamental systems of continuous semi-norms on E_1, E_2 as defined in 2.4. Then

$$\begin{aligned}\sigma(T_1 \hat{\otimes} I_2, I_1 \hat{\otimes} T_2; A_{cs}) &\supseteq \bigcap_\Gamma \sigma(T_1 \hat{\otimes} I_2, I_1 \hat{\otimes} T_2; A \cap G(E_1 \hat{\otimes}_\alpha E_2; \Gamma)) \\ &= \bigcap_{\Gamma_1} \sigma(T_1; G(E_1; \Gamma_1)) \times \bigcap_{\Gamma_2} \sigma(T_2; G(E_2; \ _2)) \\ &= \sigma(T_1; G(E_1)) \times \sigma(T_2; G(E_2)) \\ &= \sigma(T_1 \hat{\otimes} I_2, I_1 \hat{\otimes} T_2; A_{cs}).\end{aligned}$$

The first inclusion is true because of the definition of A_{cs}, the first equality is true by 2.4, the second by (0.8), and the last one is 2.1. The intersection is taken over all D-fundamental systems Γ. Therefore we obtained the following approximation theorem for joint spectra:

2.5. COROLLARY. *Let the assumptions of 2.1 be fulfilled. Then*

$$\sigma(T_1 \hat{\otimes} I_2, I_1 \hat{\otimes} T_2; A_{cs}) = \bigcap_\Gamma \sigma(T_1 \hat{\otimes} I_2, I_1 \hat{\otimes} T_2; A \cap G(E_1 \hat{\otimes}_\alpha E_2; \Gamma)),$$

where the intersection is taken over all D-fundamental systems.

2.6. Remark. We do not know, whether 2.1 holds with A_{cs} replaced by A_b. For all examples we know this is true. Moreover, taking the notations of 2.1, we pose the following problem concerning "classical" joint spectra:

Let E_1 and E_2 denote two Fréchet spaces. Prove or disprove

$$\sigma(T_1 \hat{\otimes} I_2, I_1 \hat{\otimes} T_2; A) = \sigma(T_1; L(E_1)) \times \sigma(T_2; L(E_2)).$$

3. Spectral mapping theorems. In section 2 we succeeded in describing the Cartesian product $\sigma(T_1; (L_s(E_1))_b) \times \sigma(T_2; (L_s(E_2))_b)$ as a bicommutant joint spectrum. In order to solve the announced problem, we have to establish polynomial spectral mapping theorems. For that purpose we start with a refinement of 2.5.

3.1. APPROXIMATION THEOREM. *Let E_1 and E_2 denote two locally convex spaces, and let α denote a tensor product topology fulfilling (1.3). Moreover let $T_i \in (L_s(E_i))_b$ $(i = 1,2)$, and let A be the bicommutant of $T_1 \hat{\otimes} I_2, I_1 \hat{\otimes} T_2$ in $L_s(E_1 \hat{\otimes}_\alpha E_2)$. Given $C \in A_{cb}$ and an open neighborhood U of $\sigma(T_1 \hat{\otimes} I_2, I_1 \hat{\otimes} T_2; A_{cs})$, there exists a D-fundamental system Γ on $E_1 \hat{\otimes}_\alpha E_2$ such that*

$$U \supset \sigma(T_1 \hat{\otimes} I_2, I_1 \hat{\otimes} T_2; A \cap G(E_1 \hat{\otimes}_\alpha E_2; \Gamma))$$

and

$$T_1 \hat{\otimes} I_2,\ I_1 \hat{\otimes} T_2,\ C \in A \cap G(E_1 \hat{\otimes}_\alpha E_2; \Gamma)).$$

In order to prove this, we need the following slight modification of Proposition 1.5.

3.2. PROPOSITION. *Let $G_i \subset L(E_i)$ $(i = 1,2)$ denote equicontinuous subsets such that $G_i \circ G_i \subset G_i$. Let Γ denote any D-fundamental system on $E_1 \hat{\otimes}_\alpha E_2$, and let $C \in G(E_1 \hat{\otimes}_\alpha E_2; \Gamma)$. If C commutes with all elements of $G_1 \otimes I_2$ and $I_1 \otimes G_2$, then there exists a D-fundamental system $\bar{\Gamma}$ such that*

$$C \in G(E_1 \hat{\otimes}_\alpha E_2; \bar{\Gamma}) \text{ and } \sup\{\| T \otimes S \|_{\bar{\Gamma}} : T \in G_1,\ S \in G_2\} \leq 1.$$

The D-fundamental system $\bar{\Gamma}$ constructed in the proof of 1.5 has the desired properties.

Proof of 3.1. Let $U_i \supset \sigma(T_i; (L_s(E_i))_b)$ $(i = 1,2)$ denote open neighborhoods such that $U_1 \times U_2 \subset U$. By [19], Lemma 12 there exist fundamental system Γ_i of continuous semi-norms on E_i such that

$$T_i \in G(E_i; \Gamma_i) \text{ and } \sigma(T_i; G(E_i; \Gamma_i)) \subset U_i \quad (i = 1,2).$$

Therefore the sets $\{(T_i - \lambda) : \lambda \in \mathbb{C} \setminus U_i\}$ are contained in a multiple of the unit-balls $B_1^{(i)}$ of $(G(E_i; \Gamma_i), \|\cdot\|_{\Gamma_i}) \cap \{T_i; L(E_i)\}^{cc}$.

Let $C \in G(E_1 \hat{\otimes}_\alpha E_2; \Gamma) \cap A$. Since $\{T_1; L(E_1)\}^{cc} \otimes I_2$ and $I_1 \otimes \{T_2; L(E_2)\}^{cc}$ are contained in the commutative algebra A, C and $B_1^{(i)}$ fulfill the assumptions of 3.2. Hence there exists a D-fundamental system $\overline{\Gamma}$ such that $C \in G(E_1 \hat{\otimes}_\alpha E_2;\overline{\Gamma})$ and

$$\sup\{\|T\hat{\otimes}S\|_{\overline{\Gamma}} : T\in B_1^{(1)},\ S\in B_1^{(2)}\} \leq 1.$$

But this especially means, that for all $\lambda\in\mathbb{C}\setminus U_1$ and $\gamma\in\mathbb{C}\setminus U_2$ we have

$$\|(T_1-\lambda)^{-1}\hat{\otimes} I_2\|_{\overline{\Gamma}} \leq \|(T_1-\lambda)^{-1}\|_{\Gamma_1} \quad\text{and}$$
$$\|I_1\hat{\otimes}(T_2-\gamma)^{-1}\|_{\overline{\Gamma}} \leq \|(T_2-\gamma)^{-1}\|_{\Gamma_2}.$$

But from this it follows, that every $(\lambda,\gamma)\in\mathbb{C}^2\setminus(U_1\times U_2)$ is already contained in $\rho(T_1\hat{\otimes}I_2, I_1\hat{\otimes}T_2; A\cap G(E_1\hat{\otimes}_\alpha E_2;\overline{\Gamma}))$, and hence

$$\sigma(T_1\hat{\otimes}I_2, I_1\hat{\otimes}T_2; A\cap G(E_1\hat{\otimes}_\alpha E_2;\overline{\Gamma})) \subset U_1\times U_2\subset U. //$$

Theorem 3.1 turns out to be the main tool in order to prove the second part of the following

3.3. SPECTRAL MAPPING THEOREM. *Let $T_i\in L(E_i)$ $(i = 1,2)$, let P be a polynomial in two variables, and let A denote the bicommutant of $T_1\hat{\otimes}I_2$, $I_1\hat{\otimes}T_2$ in the algebra $L_s(E_1\hat{\otimes}_\alpha E_2)$.*

(i) If $\sigma(T_i;(L_s(E_i))_b)\neq\mathbb{C}$ $(i = 1,2)$, then

$$P(\sigma(T_1;(L_s(E_1))_b),\sigma(T_2;(L_s(E_2))_b))\subset \sigma(P(T_1\hat{\otimes}I_2, I_1\hat{\otimes}T_2);A_{cb})$$

(ii) If moreover T_1 and T_2 are Allan-bounded, then

$$P(\sigma(T_1;(L_s(E_1))_b),\sigma(T_2;(L_s(E_2))_b) = \sigma(P(T_1\hat{\otimes}I_2, I_1\hat{\otimes}T_2),A_{cs})$$
$$= \sigma(P(T_1\hat{\otimes}I_2, I_1\hat{\otimes}T_2),A_{cb})$$

Proof. By 2.1 and 2.5 we have

$$\sigma(T_1;(L_s(E_1))_b)\times\sigma(T_2;(L_s(E_2))_b) =$$
$$= \bigcap_{\Gamma}\sigma(T_1\hat{\otimes}I_2, I_1\hat{\otimes}T_2; A\cap G(E_1\hat{\otimes}_\alpha E_2;\Gamma))$$

where it is sufficient to take the intersection over all those D-fundamental system Γ such that neither $\sigma(T_1\hat{\otimes}I_2; A\cap G(E_1\hat{\otimes}_\alpha E_2;\Gamma))$ nor $\sigma(I_1\hat{\otimes}T_2; A\cap G(E_1\hat{\otimes}_\alpha E_2;\Gamma))$ cover the whole complex plane. Since $(A\cap G(E_1\hat{\otimes}_\alpha E_2;\Gamma), \|\cdot\|_\Gamma)$ is a Banach algebra by (0.3), we apply Lemma 0.2 which gives

$$P(\sigma(T_1\hat{\otimes}I_2, I_1\hat{\otimes}T_2; A\cap G(E_1\hat{\otimes}_\alpha E_2;\Gamma)))\subset$$
$$\subset\sigma(P(T_1\hat{\otimes}I_2, I_1\hat{\otimes}T_2); A\cap G(E_1\hat{\otimes}_\alpha E_2;\Gamma))$$

and hence

$$P(\sigma(T_1;(L_s(E_1))_b),\sigma(T_2;(L_s(E_2))_b)) =$$
$$= P(\bigcap_{\Gamma}\sigma(T_1\hat{\otimes}I_2, I_1\hat{\otimes}T_2; A\cap G(E_1\hat{\otimes}_\alpha E_2;\Gamma)))$$
$$\subseteq\bigcap_{\Gamma}P(\sigma(T_1\hat{\otimes}I_2, I_1\hat{\otimes}T_2; A\cap G(E_1\hat{\otimes}_\alpha E_2;\Gamma)))$$

$$\subset \bigcap_{\Gamma} \sigma(P(T_1\hat{\otimes} I_2, I_1\hat{\otimes} T_2); A\cap G(E_1 \hat{\otimes}_\alpha E_2;\Gamma)) =$$
$$= \sigma(P(T_1\hat{\otimes} I_2, I_1\hat{\otimes} T_2); A_{cb}).$$

This proves (i).
In order to prove (ii), let $\gamma \notin P(\sigma(T_1;(L_s(E_1))_b), \sigma(T_2;(L_s(E_2))_b))$ and let V denote an open neighborhood of this set such that $\gamma \notin V$. Then there exists an open neighborhood $U \supset \sigma(T_1\hat{\otimes} I_2, I_1\hat{\otimes} T_2; A_{cs})$ such that $P(U) \subset V$. By 3.1 there exists a D-fundamental system Γ such that $T_1\hat{\otimes} I_2$, $I_1\hat{\otimes} T_2 \in G(E_1 \hat{\otimes}_\alpha E_2;\Gamma)$ and U contains $\sigma(T_1\hat{\otimes} I_2, I_1\hat{\otimes} T_2; A\cap G(E_1 \hat{\otimes}_\alpha E_2;\Gamma))$, and hence $\gamma \notin P(\sigma(T_1\hat{\otimes} I_2, I_1\hat{\otimes} T_2; A\cap G(E_1 \hat{\otimes}_\alpha E_2;\Gamma)))$. Since $T_1\hat{\otimes} I_2$, $I_1\hat{\otimes} T_2$ are elements of the Banach algebra $(A\cap G(E_1 \hat{\otimes}_\alpha E_2;\Gamma), \|\cdot\|_\Gamma)$, the spectral mapping theorem for joint spectra in Banach algebras gives

$$(*) \quad \begin{aligned} &P(\sigma(T_1\hat{\otimes} I_2, I_1\hat{\otimes} T_2; A\cap G(E_1 \hat{\otimes}_\alpha E_2;\Gamma))) = \\ &\quad = \sigma(P(T_1\hat{\otimes} I_2, I_1\hat{\otimes} T_2); A\cap G(E_1 \hat{\otimes}_\alpha E_2;\Gamma)). \end{aligned}$$

But this implies $\gamma \notin \sigma(P(T_1\hat{\otimes} I_2, I_1\hat{\otimes} T_2); A_{cb})$. It remains to be shown that

$$C := (\gamma I_1\hat{\otimes} I_2 - P(T_1\hat{\otimes} I_2, I_1\hat{\otimes} T_2))^{-1} \in A_{cs}.$$

Thus let $D \in (\{C; L_s(E_1 \hat{\otimes}_\alpha E_2)\}^{cc})_{cb}$. In applying 3.1 we find a D-fundamental system $\bar{\Gamma}$ such that $D, T_1\hat{\otimes} I_2, I_1\hat{\otimes} T_2 \in G(E_1 \hat{\otimes}_\alpha E_2;\bar{\Gamma})$ and $U \supset \sigma(T_1\hat{\otimes} I_2, I_1\hat{\otimes} T_2; A\cap G(E_1 \hat{\otimes}_\alpha E_2;\bar{\Gamma}))$. A repetion of the spectral mapping theorem (*) gives $\gamma \notin \sigma(P(T_1\hat{\otimes} I_2, I_1\hat{\otimes} T_2); A\cap G(E_1 \hat{\otimes}_\alpha E_2;\bar{\Gamma}))$. But this means, that $C \in A\cap G(E_1 \hat{\otimes}_\alpha E_2;\bar{\Gamma})$, and hence $C \in A_{cs}$. Connected with part (i) this proves (ii).//

3.4. Remarks.
(1) We do not know whether 3.1 (ii) can be sharpened by substituting A_b for A_{cb}. Such a result has been announced by Kawamura [18] in the case where E_1 and E_2 are Fréchet spaces, one of which has to be nuclear, but it turns out that there is a gap in Kawamura's proof, because the central Proposition 4.1 is false.
(2) The spectral mapping theorem can be generalized from polynomials to functions, which are analytic in a neighborhood of the joint spectrum, by means of an analytic functional calculus (see [28], II.4 for details).
(3) The inclusion in 3.3 (i) is strict in general as we shall illustrate by several counterexamples below. In order to get the reverse inclusion for non-Allan-bounded operators, too, one has to pose additional conditions upon the operators T_1, T_2 and/or the polynomial P. In [28], III. we have given a functional calculus ori-

ginally due to Sebastião e Silva [26], which guarantees two-sided spectral mapping theorems under conditions that are fulfilled in applications to abstract Cauchy problems. We shall give an example below.

3.5. COUNTEREXAMPLES.

For $n \in \mathbb{Z}$ let

$$\mathcal{E}_{(-\infty,n)} := \{u \in C^\infty(\mathbb{R}) : \text{supp } u \subset (-\infty,n)\}$$

equipped with the topology induced from $C^\infty(\mathbb{R})$. Then the space

$$\mathcal{D}_- := \bigcup_{n\in\mathbb{Z}} \mathcal{E}_{(-\infty,n)}$$

being equipped with the canonical topology of the inductive limit is a strict (LF)-space, and a nuclear space. Hence the strong dual

$$\mathcal{D}'_+ := (\mathcal{D}_-)'_s$$

is especially a Montel space, and hence $L_s(\mathcal{D}'_+)$ is sequentially complete. As sets we have

$$\mathcal{D}'_+ = \{\phi \in \mathcal{D}' : \text{supp } \phi \text{ is bounded from the left}\}$$

and hence $\mathcal{D}'_+$ admits a convolution product. For every $\lambda \in \mathbb{C}$ the operator $\frac{d}{dx} - \lambda$ has an inverse in $L(\mathcal{D}'_+)$, which can be written as convolution operator $(e^{\lambda\cdot}H(.))*$. Consequently the spectrum $\sigma(\frac{d}{dx};L(\mathcal{D}'_+))$ is empty, but since the function $\lambda \rightsquigarrow (e^{\lambda\cdot}H(.))*$ is analytic with respect to the $L_s(\mathcal{D}'_+)$-topology, $\sigma(\frac{d}{dx};(L_s(\mathcal{D}'_+))_b)$ is empty, too, and hence $\sigma((\frac{d}{dx})^{-1};(L_s(\mathcal{D}'_+))_b) = \{0\}$.

A necessary (but by no means sufficient) condition for a distribution $u \in \mathcal{D}'(\mathbb{R}^2)$ to be contained in $\mathcal{D}'_+ \hat{\otimes}_\pi \mathcal{D}'_+$ is, that for every $\phi \in \mathcal{D}_-$, $u(\phi)$ belongs to $\mathcal{D}'_+$ (look at u as an element from $L(\mathcal{D}_-, \mathcal{D}'_+)$; Schwartz [24], p. 51).

Next let $v \in C^\infty(\mathbb{R}^2 \setminus \{0\})$ denote a function, which is real-analytic outside the origin, and let $\phi \in C_0^\infty(\mathbb{R} \setminus \{0\})$. Then the function

(*) $\qquad y \rightsquigarrow \int_{-\infty}^{\infty} v(x,y)\phi(x)dx$

is real-analytic, and hence is contained in $\mathcal{D}'_+$ if and only if it vanishes identically. From these facts we infer, that the Cauchy-Riemann operator

$$\frac{d}{dx}\hat{\otimes} I + iI\hat{\otimes}\frac{d}{dy} : \mathcal{D}'_+ \hat{\otimes}_\pi \mathcal{D}'_+ \longrightarrow \mathcal{D}'_+ \hat{\otimes}_\pi \mathcal{D}'_+$$

has no fundamental solution in $\mathcal{D}'_+ \hat{\otimes}_\pi \mathcal{D}'_+$, and consequently is not surjective. But this is clear, since every fundamental solution of the Cauchy-Riemann operator is analytic outside the origin, and hence by (*) cannot be contained in $\mathcal{D}'_+ \hat{\otimes}_\pi \mathcal{D}'_+$.

After these preliminary remarks we shall consider three cases, which show, that for non-Allan-bounded operators the spectral mapping theorem 3.3 (i) can fail in a dramatic manner.

(1) Let $E_1 = E_2 = \mathcal{D}'_+$, let $T_1 := \frac{d}{dx}$, $T_2 := i\frac{d}{dy}$. Then $\sigma(T_i;(L_s(E_i))_b) = \emptyset$ $(i = 1,2)$. On the other hand, we shall show, that

$$\sigma(T_1\hat{\otimes} I_2 + I_1\hat{\otimes} T_2; L(E_1\hat{\otimes}_\pi E_2)) = \mathbb{C}.$$

For that purpose note that $T_1\hat{\otimes} I_2 + I_1\hat{\otimes} T_2 - \lambda I_1\hat{\otimes} I_2$ $(\lambda\in\mathbb{C})$ has no fundamental solution u in $E_1\hat{\otimes}_\pi E_2$, for otherwise $\exp(-\frac{\lambda}{2}(x - iy))u$ would be a fundamental solution of the Cauchy-Riemann operator.

(2) Let E_1, E_2, and T_1 as above, and let $T_2 := (\frac{d}{dy})^{-1}$. Then by similar arguments as above, one can prove (cf. [28], II.4.18)

$$\sigma(T_1\hat{\otimes} T_2;(L_s(E_1\hat{\otimes}_\pi E_2))_b) = \mathbb{C}.$$

(3) Let $E_1 := \prod_1^\infty \mathbb{C}$, $T_1\in L(E_1)$ the left-shift, $E_2 = \mathcal{D}'_+$, $T_2 = \frac{d}{dy}$. Then $T_1\hat{\otimes} I_2$ operates as left-shift on $\prod_1^\infty E_2 \cong E_1\hat{\otimes}_\pi E_2$, and $I_1\hat{\otimes} T_2$ operates coordinate-wise. Thus for every $\lambda\in\mathbb{C}$ the sequence $((\lambda T_2^{-1})\delta)_{n\in\mathbb{N}}$ is an eigenvector of $T_1\hat{\otimes} T_2$, and hence

$$\sigma(T_1\hat{\otimes} T_2; L(E_1\hat{\otimes}_\pi E_2)) = \mathbb{C}.$$

4. <u>A distributional Cauchy problem</u>. Since it is beyond the scope of this paper to present the functional calculus as devellopped in [28], we content ourselves with giving a typical example illustrating of what kind conditions have to be assuring a tensor product of operators to be invertible. For that purpose let $T\in L(E)$, let P denote a polynomial in two variables, and consider the distributional Cauchy problem for the operator

$$P(\tfrac{d}{dt}\hat{\otimes} I, I\hat{\otimes} T) : \mathcal{D}'_+\hat{\otimes}_\pi E \longrightarrow \mathcal{D}'_+\hat{\otimes}_\pi E$$

If $P(z_1,z_2) = z_1 - z_2$ this is a tensor product notation of a Hille-Yosida-type problem.

In order to give sufficient conditions for $P(\frac{d}{dt}\hat{\otimes} I, I\hat{\otimes} T)$ to be invertible, we need the following (cf. [28], p. 60)

4.1. LEMMA. *For all* $k\in\mathbb{N}$ *the function*

$$\lambda \rightsquigarrow (\tfrac{d}{dt} - \lambda)^{-1} = e^{\lambda\cdot}H(.)*\in L_s(\mathcal{D}'_+)$$

is bounded on $F_k = \{\lambda\in\mathbb{C} : Re(\lambda) < k\}$.

4.2. THEOREM. *Let* E *be a locally convex space, let* $T\in L(E)$, *and let* $c\in\mathbb{R}$. *Suppose that the function* $\lambda \rightsquigarrow P(\lambda, T)^{-1}$ *is bounded on*

$\mathbb{C} \setminus F_c = \{\lambda \in \mathbb{C} : Re(\lambda) \geq c\}$. Then $P(\frac{d}{dt}\hat{\otimes} I, I\hat{\otimes} T) \in L(\mathcal{D}'_+ \hat{\otimes}_\pi E)$ is invertible and

$$P(\frac{d}{dt}\hat{\otimes} I, I\hat{\otimes} T)^{-1} = (\frac{d}{dt} - \lambda_o)^2 \hat{\otimes} I \frac{1}{2\pi i} \int_{c+i\mathbb{R}} (\lambda-\lambda_o)^{-2} (\frac{d}{dt} - \lambda)^{-1} \hat{\otimes} P(\lambda, T)^{-1} d\lambda,$$

where $\lambda_o \in F_c$ is arbitrarily chosen and the integral is an improper Riemann integral.

We sketch a proof. By 4.1 the function $\lambda \rightsquigarrow (\frac{d}{dt} - \lambda)^{-1} \in L_s(\mathcal{D}'_+)$ is bounded on F_c. Therefore the integral exists as an improper Riemann-integral. Multiplying this integral by $P(\frac{d}{dt}\hat{\otimes} I, I\hat{\otimes} T)$ we obtain

$$P(\frac{d}{dt}\hat{\otimes} I, I\hat{\otimes} T) \int_{c+i\mathbb{R}} \ldots = \int \{(P(\frac{d}{dt}\hat{\otimes} I, I\hat{\otimes} T) - P(\lambda I\hat{\otimes} I, I\hat{\otimes} T)\} \ldots d\lambda + \int P(\lambda I\hat{\otimes} I, I\hat{\otimes} T) \ldots d\lambda,$$

where ... denotes the integrand of the integral in the statement of the theorem. In the first integrand, { } contains $(\frac{d}{dt} - \lambda)\hat{\otimes} I$ as a factor. Therefore the first integral vanishes by a residuum calculation. Since $P(\lambda I\hat{\otimes} I, I\hat{\otimes} T) \bullet I\hat{\otimes} P(\lambda, T)^{-1} = I\hat{\otimes} I$, the second integral gives $2\pi i(\frac{d}{dt} - \lambda_o)^{-2}\hat{\otimes} I$, which proves the theorem.

By taking an m-th power of $(\lambda-\lambda_o)^{-1}$ and $(\frac{d}{dt} - \lambda_o)$ if necessary, the assumptions of the theorem can be weakened, so that $P(.,T)^{-1}$ is polynomially increasing in some right half-plane. It is well known, that in such a case $P(.,T)^{-1}$ is Laplace-transform of an operator-valued distribution u. In the case of a Banach space E, u can be shown to be a fundamental solution for the operator $P(\frac{d}{dt}\hat{\otimes} I, I\hat{\otimes} T)$, and one gets the inverse operator by means of convolution of vector-valued distributions. This method of proof for 4.2 is the so-called Laplace-transform-method (cf. Beals [2]). It seems to us, that our proof is more elementary even for Banach spaces. If E is a proper locally convex space, the Laplace-transform does not work in general, because a convolution cannot be defined on the whole of $\mathcal{D}'_+ \hat{\otimes}_\pi E$ (cf. Fattorini [7]).

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FEDERAL REPUBLIC GERMANY