

L. Tamássy; Bela Kis

Relations between Finsler and affine connections

In: Zdeněk Frolík (ed.): Proceedings of the 11th Winter School on Abstract Analysis. Circolo Matematico di Palermo, Palermo, 1984. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 3. pp. [329]--337.

Persistent URL: <http://dml.cz/dmlcz/701323>

Terms of use:

© Circolo Matematico di Palermo, 1984

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.



This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* <http://project.dml.cz>

RELATIONS BETWEEN FINSLER AND AFFINE CONNECTIONS

L. Tamássy - B. Kis

We consider a Finsler connection Γ_F as a linear connection in the vertical subbundle $V\mathcal{CB}$ of $\tau\mathcal{CB}$, β denoting a differentiable manifold, \mathcal{CB} its tangent bundle, and $\tau\mathcal{CB}$ the total space of \mathcal{CB} .^{1/} Of course such a Finsler connection can be extended in many ways to a linear connection Γ in $\tau\mathcal{CB}$. V. Oproiu [3] gave a way of such an extension in which the connection in $\tau\mathcal{CB}$ is uniquely determined by the Finsler connection.

In this paper we investigate another way of extension using beside the Finsler connection a connection $\bar{\Gamma}$ in \mathcal{CB} . Moreover we investigate the inverse problem: given a linear connection Γ in $\tau\mathcal{CB}$, when is this an extension in our above sense of a Finsler connection Γ_F , and what is the connection $\bar{\Gamma}$ used at this extension. Thus we touch upon the question, when does the restriction to the vertical subspace $V\mathcal{CB}$ of a linear connection Γ in $\tau\mathcal{CB}$ yield a Finsler connection.

In §1. we collect and partially supplement the notions and tools used in our investigations. Whitney sums and Whitney decompositions of connections have an important role throughout in our investigations. §2 deals with the extensions, and mainly with the mentioned inverse problem leading to a system of partial differential equations whose integrability is also investigated.

§1. Connections, Whitney sums and Whitney decompositions

1. Bundles. If $\mathcal{Q} = (M, N, \pi)$ is a bundle, then $\tau\mathcal{Q} \doteq M$ denotes the total space of \mathcal{Q} , $b\mathcal{Q} \doteq N$ is the base space of \mathcal{Q} , and $\pi\mathcal{Q} \doteq \pi: M \rightarrow N$ the projection map. Manifolds, bundles and maps are supposed to be of class C^∞ throughout the paper if not otherwise stated. If φ is map of the bundle \mathcal{Q} on the bundle \mathcal{Q}

1/ There are several slightly different definitions a Finsler connection. In this paper we use the above one.

, then ξ is the domain of ψ , $\text{dom } \psi = \xi$. If the diagram

$$\begin{array}{ccc} \text{tl } \xi & \xrightarrow{\text{tl } \psi} & \text{tl } \eta \\ \text{pr } \xi \downarrow & & \downarrow \text{pr } \eta \\ \text{bs } \xi & \xrightarrow{\text{bs } \psi} & \text{bs } \eta \end{array}$$

is commutative $\text{tl } \psi$ and $\text{bs } \psi$ denoting the map showed by the figure, then ψ is called a bundle map.

$\mathcal{T}M$ denotes the tangent bundle of the manifold M , and we also use the notations $\mathcal{T}M = \text{tl } \mathcal{T}M$, $\mathcal{V}_M = \text{pr } \mathcal{T}M$. The tangent bundle of a bundle ξ is $\mathcal{T}\xi = (\mathcal{T}\text{tl}\xi, \mathcal{T}\text{pr}\xi, \mathcal{T}\text{bs}\xi)$ where $\mathcal{T}\text{pr}\xi$ denotes the differential of the map $\text{pr}\xi$ denoted sometimes also by $d\text{pr}\xi$. $\mathcal{V}\xi = (\text{Ker } \mathcal{T}\text{pr}\xi, \mathcal{V}\text{tl}\xi, \text{tl}\xi)$ is called the vertical subbundle of $\mathcal{T}\xi$.

Let ξ be a bundle, M a differentiable manifold, and $\psi: M \rightarrow \text{bs}\xi$ a map. If the elements α_1, α_2 of the set $\Pi(\xi, \psi) = \{\alpha: \mathcal{V} \rightarrow \xi \mid \alpha \text{ is a bundle map, } \text{bs}\alpha = \psi\}$ satisfy the property that there exists a bundle map $\sigma: \text{dom } \alpha_1 \rightarrow \text{dom } \alpha_2$ for which $\alpha_1 = \alpha_2 \circ \sigma$ then we say that α_2 factorizes α_1 .

σ will be denoted by $\text{Fact}_\xi(\alpha_1, \alpha_2)$. The unique element of the set $\Pi(\xi, \psi)$ factorizing every element of $\Pi(\xi, \psi)$ is denoted by $\text{ad}_\xi \psi$, and $\text{Fact}_\xi(\alpha, \text{ad}_\xi \psi)$ by $\text{Fact}_\xi \alpha$. $\text{dom } \text{ad}_\xi \psi$ is called the pull-back bundle of ξ by ψ and we denote it by $\psi^! \xi$. It is well known that $\mathcal{V}\xi$ and $(\text{pr}\xi)^! \xi$ are canonically isomorphic. We denote it by

(1)
$$i_\xi: (\text{pr}\xi)^! \xi \rightarrow \mathcal{V}\xi$$

A vector bundle ξ is the Whitney sum of the vector bundles ξ^1, ξ^2 ; $\text{bs}\xi^1 = \text{bs}\xi^2$ if there exist bundle homomorphisms $\rho^1: \xi \rightarrow \xi^1, \rho^2: \xi \rightarrow \xi^2$ ($\rho = \rho^1 + \rho^2$) satisfying the conditions:

(2)
$$\rho^1 \circ \rho^1 = \begin{cases} \phi & \text{if } \nu \neq \mu \\ \text{id}_\xi & \text{if } \nu = \mu \end{cases} \text{ and } \rho^1 \circ \rho^1 + \rho^2 \circ \rho^2 = \text{id}_\xi$$

As it is well known, the sequences $0 \rightarrow \xi^1 \xrightarrow{\rho^1} \xi \xrightarrow{\rho^2} \xi^2 \rightarrow 0$ and $0 \leftarrow \xi^1 \xleftarrow{\rho^1} \xi \xleftarrow{\rho^2} \xi^2 \leftarrow 0$ are short exact sequences, and the corresponding maps split each other. We indicate this fact by the diagram

(3)
$$0 \leftarrow \xi^1 \xleftarrow{\rho^1} \xi \xleftarrow{\rho^2} \xi^2 \leftarrow 0$$

(3) obviously determines a Whitney sum, and we will give Whitney sums mostly in this form, and say that (3) is a dual short exact sequence. Let ξ^1 and ξ^2 be subbundles of ξ such that every fibre of ξ over an $x \in \text{bs}\xi$ is the direct sum of the appropriate

fibres of ξ^1 and ξ^2 . Furthermore let $\rho^v: \xi \rightarrow \xi^v$ and be the natural projections and inclusions. Then (3) is a dual short exact sequence. In this case we write $\xi = \xi^1 \oplus \xi^2$. This is the classical Whitney sum of ξ^1 and ξ^2 . - In the subsequent part of the paper bundles are always vector bundles.

2. Connections. A connection Γ on a vector bundle ξ is the dual short exact sequence

$$(4) \quad 0 \longleftarrow (\mathcal{P}\mathcal{R}\xi)^! \xi \xleftarrow[\mathcal{J}_\xi^U]{U_\xi} \mathcal{C}\mathcal{L}\xi \xrightarrow[\mathcal{R}_\xi]{h_\xi} (\mathcal{P}\mathcal{R}\xi)^! \mathcal{C}\mathcal{L}\xi \longleftarrow 0$$

where \mathcal{J}_ξ^U is the inclusion map induced by i_ξ , and $\mathcal{R}_\xi \doteq \text{Fact}_{\mathcal{C}\mathcal{L}\xi} d(\mathcal{P}\mathcal{R}\xi)$. The Dombrovski map of the connection Γ is

$$(5) \quad K_\Gamma \doteq \text{ad}_\xi \mathcal{P}\mathcal{R}\xi \circ U_\xi : \mathcal{C}\mathcal{L}\xi \rightarrow \xi$$

The covariant derivative ∇ associated to Γ is

$$(6) \quad \nabla(U_i \sigma) \doteq (K_\Gamma \circ T\sigma)$$

A connection is uniquely determined by its Dombrovski map.

Proposition 1. The bundle homomorphism $K: \mathcal{C}\mathcal{L}\xi \rightarrow \xi$ is the Dombrovski map of some connection Γ iff

$$(7) \quad K|_{V_\xi} = \text{ad}_\xi \mathcal{P}\mathcal{R}\xi \circ \mathcal{V}_\xi$$

where $\mathcal{V}_\xi = i_\xi^{-1}$.

Proof. Suppose that K is the Dombrovski map of a connection (4). Then $K|_{V_\xi} = (\text{ad}_\xi \mathcal{P}\mathcal{R}\xi \circ U_\xi)|_{V_\xi} = \text{ad}_\xi \mathcal{P}\mathcal{R}\xi \circ (U_\xi)|_{V_\xi} = \text{ad}_\xi \mathcal{P}\mathcal{R}\xi \circ \mathcal{V}_\xi$ because of $U_\xi \circ \mathcal{J}_\xi^U = \text{id}_{(\mathcal{P}\mathcal{R}\xi)^! \xi}$, and $\mathcal{J}_m \mathcal{J}_\xi^U = V_\xi$.

Conversely, suppose that the bundle homomorphism $K: \mathcal{C}\mathcal{L}\xi \rightarrow \xi$ satisfies the condition (7). Now $\text{Ker } K = \text{Ker } U_\xi$ because $\text{ad}_\xi \mathcal{P}\mathcal{R}\xi$ is a bijection on the fibres. Moreover $K(V_\xi) = (\text{ad}_\xi \mathcal{P}\mathcal{R}\xi \circ \mathcal{V}_\xi)(V_\xi) = (\text{ad}_\xi \mathcal{P}\mathcal{R}\xi)((\mathcal{P}\mathcal{R}\xi)^! \xi) = \xi$, thus $V_\xi \oplus \text{Ker } U_\xi = \mathcal{C}\mathcal{L}\xi$ therefore we can write every $z \in \mathcal{C}\mathcal{L}\xi$ in the form $z = x + y$, where $x \in V_\xi$ and $y \in \text{Ker } U_\xi = \text{Ker } K$. Now, $U_\xi(z) = U_\xi(x) + 0 = (U_\xi|_{V_\xi})(x) = \mathcal{V}_\xi(x)$, which has already been given. Thus $U_\xi(z)$ is given if we know $\text{Ker } K$. On the other hand U_ξ uniquely determines the connection (4).

Let ξ^1 be a vector-subbundle of ξ . We say that the connection Γ is invariant on the subbundle ξ^1 , if

$$\nabla(U_i \sigma) \in \text{dec } \xi^1 \text{ for every } U_i \in \text{dec } \mathcal{C}\mathcal{L}\xi \text{ and for every } \sigma \in \text{dec } \xi^1,$$

i.e. if $(K_\Gamma \circ T\sigma)(U_i) \in \xi^1$. This is equivalent with the property

$$[(K_\Gamma \circ T)(U_i)](U_j) \in \text{dec } \xi^1 \text{ for every } U_j \in \mathcal{C}\mathcal{L}\xi, U_i \in \text{dec } \xi^1.$$

Let $\alpha: \xi \rightarrow \xi'$ be an isomorphism between the bundles ξ and ξ' . Now a connection Γ on ξ induces a connection Γ' on ξ' by the Dombrovski map $K_{\Gamma'} \doteq \alpha \circ K_{\Gamma} \circ (d\alpha^{-1})$

We denote $\mathcal{J}_m \xi$ by $H\xi$. $H\xi$ uniquely determines the connection (4). The map P_{ξ} is a surjection /on the fibres/, and its kernel consists of the fibres of $V\xi$. Thus P_{ξ} is a bijection on $H\xi$, and there exists the inverse of $P_{\xi}|_{H\xi}$. Denote this inverse of $P_{\xi}|_{H\xi}$ by h_{ξ} . The map \tilde{P}_{ξ} is an injection /on the fibres/ and its image is $V\xi$. We know that $j_{\xi} = (P_{\Gamma\xi})^{-1} \xi \rightarrow V\xi$ is a canonical isomorphism, and the inverse of j_{ξ} is denoted by τ_{ξ} .

3. Whitney sums and Whitney decompositions of a connection. Let ξ_1, ξ_2 ($\nu=1,2$) be vector bundles, satisfying (3), thus ξ the Whitney sum of ξ^{ν} . Let a connection Γ^{ν} be given on ξ^{ν} :

$$(8) \quad 0 \longleftarrow (P_{\Gamma\xi^{\nu}})^{-1} \xi^{\nu} \xrightarrow[\tilde{P}_{\xi^{\nu}}]{U_{\xi^{\nu}}} \tau_{\xi} \xi \xrightarrow[\tilde{P}_{\xi}]{h_{\xi^{\nu}}} (P_{\Gamma\xi^{\nu}})^{-1} \xi^{\nu} \longrightarrow 0$$

and let $K_{\Gamma^{\nu}}$ be the Dombrovski map of Γ^{ν} . Then we can give a connection Γ on the vector bundle ξ with the aid of Γ^{ν} and (3). The simplest way of giving this connection is using its Dombrovski map, defined by

$$(9) \quad K_{\Gamma} = U^1 \circ K_{\Gamma^1} \circ d\rho^1 + U^2 \circ K_{\Gamma^2} \circ d\rho^2$$

With the help of Proposition 1. we can easily see that K_{Γ} is in fact a Dombrovski map of some connection Γ on ξ . Moreover, if Γ^1 and Γ^2 are linear then Γ too is so. This connection is the Whitney sum of Γ^1 and Γ^2 by (3). Conversely, let Γ be a connection on the above vector bundle ξ . Then two new connections Γ^1 and Γ^2 can be given on the vector bundles ξ^1 and ξ^2 with the aid of (3). These connections are given by their Dombrovski maps K_{Γ^1} and K_{Γ^2} :

$$(10) \quad K_{\Gamma^{\nu}} = \rho^{\nu} \circ K_{\Gamma} \circ d\rho^{\nu}$$

Γ^{ν} are called the Whitney decomposition of the connection Γ by (3) on the vector bundle ξ^{ν} . If Γ is linear, then Γ^1 and Γ^2 are linear, too.

We say that a connection Γ on ξ is invariant by (3) if the Whitney sum by (3) of the Whitney decompositions of Γ by (3) is Γ again.

Proposition 2. A connection Γ on the Whitney sum ξ of ξ^1 and ξ^2 according to (3) is invariant by (3) iff Γ is invariant on the bundles $\mathcal{J}_m \xi^{\nu}$ ($\nu=1,2$).

Proof. For the necessity we get the following equation according to the formulas (9) and (10):

$$(11) \quad K_{\sigma} = (\iota^{\circ} \rho^{\iota}) \circ K_{\sigma} \circ d(\iota^{\circ} \rho^{\iota}) + (\iota^{\circ} \rho^{\iota}) \circ K_{\sigma} \circ d(\iota^{\circ} \rho^{\iota})$$

Multiplying by ρ^{ι} and arranging our formulas we get the following equation:

$$\rho^{\iota} \circ K_{\sigma} - \rho^{\iota} \circ K_{\sigma} \circ d(\iota^{\circ} \rho^{\iota}) = \rho^{\iota} \circ K_{\sigma} \circ d(\iota^{\circ} \rho^{\iota}) - \rho^{\iota} \circ K_{\sigma} \circ d(\iota^{\circ} \rho^{\iota}) = 0$$

Thus for all $\nu \in \mathcal{C}^{\infty}(\mathcal{B})$ and for all $\sigma \in \text{Sec } \mathcal{F}_m \mathcal{L}^{\iota}$ $K_{\sigma} \circ d\sigma \in \text{Ker } \rho^{\iota} = \mathcal{F}_m \mathcal{L}^{\iota}$, and so Γ is invariant on $\mathcal{F}_m \mathcal{L}^{\iota}$. Similarly Γ is invariant on $\mathcal{F}_m \mathcal{L}^{\iota'}$, too.

The sufficiency means that

$$((\iota^{\circ} \rho^{\iota} \circ K_{\sigma} \circ d\iota^{\circ} \rho^{\iota} + \iota^{\circ} \rho^{\iota} \circ K_{\sigma} \circ d\iota^{\circ} \rho^{\iota}) \circ d\sigma)(\nu) \in \xi^{\nu}$$

for $\sigma \in \text{Sec } \mathcal{F}_m \mathcal{L}^{\iota}$ if $(K_{\sigma} \circ d\sigma)(\nu) \in \xi^{\nu}$. But this is trivial.

§2. Finsler connections and vertical invariant connections

1. A linear connection in the vector bundle $(\rho\tau\mathcal{C}\mathcal{B})^{\iota}\mathcal{C}\mathcal{B}$ or in $\nu\mathcal{C}\mathcal{B}$ is called a Finsler connection over the manifold \mathcal{B} , or on the bundle $\mathcal{C}\mathcal{B}$. Let a connection $\bar{\Gamma}$

$$(12) \quad 0 \longleftarrow (\rho\tau\mathcal{C}\mathcal{B})^{\iota}\mathcal{C}\mathcal{B} \xrightarrow[\bar{j}_{\mathcal{C}\mathcal{B}}]{\iota_{\mathcal{C}\mathcal{B}}} \mathcal{C}\mathcal{B} \xleftarrow[\bar{h}_{\mathcal{C}\mathcal{B}}]{h_{\mathcal{C}\mathcal{B}}} (\rho\tau\mathcal{C}\mathcal{B})^{\iota}\mathcal{C}\mathcal{B} \longrightarrow 0$$

be given in the vector bundle $\mathcal{C}\mathcal{B}$. Since $\mathcal{C}\mathcal{B}$ is a vector bundle \mathcal{B} we can apply the results of §1. So according to section 1.1 and formula (1), $\bar{j}_{\mathcal{C}\mathcal{B}}$ performs an isomorphism $(\rho\tau\mathcal{C}\mathcal{B})^{\iota}\mathcal{C}\mathcal{B} \rightarrow \nu\mathcal{C}\mathcal{B}$ and given a Finsler connection Γ_F in $(\rho\tau\mathcal{C}\mathcal{B})^{\iota}\mathcal{C}\mathcal{B}$ $\bar{j}_{\mathcal{C}\mathcal{B}}$ induces a connection $\hat{\Gamma}$ in $\nu\mathcal{C}\mathcal{B}$ according to section 1.2. Similarly $h_{\mathcal{C}\mathcal{B}}$ transplants the connection Γ_F in $(\rho\tau\mathcal{C}\mathcal{B})^{\iota}\mathcal{C}\mathcal{B}$ into a connection $\hat{\Gamma}$ in $\mathcal{H}\mathcal{C}\mathcal{B}$ /see section 1.2./. Then we can form the Whitney sum Γ of $\hat{\Gamma}$ with itself by (12). This yields the same as the classical Whitney sum of $\hat{\Gamma}$ and $\hat{\Gamma}$, in $\mathcal{C}\mathcal{B}$ for $\mathcal{C}\mathcal{B} = \nu\mathcal{C}\mathcal{B} \oplus \mathcal{H}\mathcal{C}\mathcal{B}$. This Γ is induced by Γ_F and $\bar{\Gamma}$, and is called the extension of the Finsler connection $\hat{\Gamma}$ with the aid of $\bar{\Gamma}$. The connection Γ determined in this way is obviously invariant on $\nu\mathcal{C}\mathcal{B}$ and $\mathcal{H}\mathcal{C}\mathcal{B}$.

Now we wish to investigate the converse problem: Is every vertical invariant linear connection Γ in $\mathcal{C}\mathcal{B}$ an extension of a Finsler connection Γ_F with the aid of an appropriate $\bar{\Gamma}$? The answer is negative. We will determine the conditions for the positive answer, and this will show that in cases when these conditions are not fulfilled, Γ is no extension of Γ_F by a $\bar{\Gamma}$.

2. A linear connection in $\tau\pi\tau\mathcal{B}$ is called a vertical invariant connection, if it is invariant on the subbundle $\nu\mathcal{B}$. It is clear that every Finsler connection in $\nu\mathcal{B}$ can be extended in many ways to a vertical invariant connection in $\tau\pi\tau\mathcal{B}$. The simplest example of such an extension was given in the previous paragraph.

From this point on we will compute locally. Let $(x^i, y^i)_{i,j=1,\dots,n}$ be the natural local coordinate system in $\tau\mathcal{B}$, determined by a local coordinate system (x^i) of \mathcal{B} . Then $(\frac{\partial}{\partial x^i}; \frac{\partial}{\partial y^i})$ is at (x, y) a local base of the fibre of $\tau\pi\tau\mathcal{B}$, and $(\frac{\partial}{\partial y^i})$ is the canonical base of $\nu\mathcal{B}$ at the same point. Be $e_i = \tau\pi\tau\mathcal{B} \frac{\partial}{\partial y^i}$ then (e_i) is a base in the fibres of the bundle $(\pi\tau\mathcal{B})^*\tau\mathcal{B}$.

The covariant derivative, associated to Γ_F is

$$(13) \quad \begin{aligned} \nabla_F \frac{\partial}{\partial x^i} &= G_{\alpha i}^k dz^\alpha \frac{\partial}{\partial x^k} + H_{\alpha i}^k dz^\alpha \frac{\partial}{\partial y^k} \\ \nabla_F \frac{\partial}{\partial y^i} &= C_{\alpha i}^k dz^\alpha \frac{\partial}{\partial y^k} \end{aligned} \quad \alpha = \overline{1, 2n}$$

since Γ_F is invariant on $\nu\mathcal{B}$. Here $z^\alpha = x^\alpha$ if $1 \leq \alpha \leq n$ and $z^\alpha = y^{\alpha-n}$ if $n+1 \leq \alpha \leq 2n$. Thus Greek indices run from 1 to $2n$, Latin indices run from 1 to n , and the summation convention is also applied.

Now we are going to express in the local bases $(\frac{\partial}{\partial x^i}; \frac{\partial}{\partial y^i})$ the covariant derivatives $\overset{\#}{\nabla} \frac{\partial}{\partial x^i}$ and $\overset{\#}{\nabla} \frac{\partial}{\partial y^i}$ associated to a connection $\overset{\#}{\Gamma}$ which is an extension of a Finsler connection $\overset{\#}{\Gamma}$ in $\nu\mathcal{B}$ with the aid of an arbitrary $\bar{\pi}$, π^1 being the transplant by $\overset{\#}{\Gamma}$ of an arbitrary Finsler connection Γ_F in $(\pi\tau\mathcal{B})^*\tau\mathcal{B}$:

$K_{\overset{\#}{\Gamma}} = \overset{\#}{\Gamma} \circ K_{\pi^1} \circ (d\overset{\#}{\Gamma})^{-1}$ /see section 1.2/. Thus $\overset{\#}{\Gamma}$ is the Whitney sum of Γ_F with itself by (12). It is clear that this construction yields all those vertical invariant connections which are extensions in the above sense. Then a comparison of $\overset{\#}{\nabla} \frac{\partial}{\partial x^i}$ and $\overset{\#}{\nabla} \frac{\partial}{\partial y^i}$ with (13) yields the sought for our conditions. First we study the maps of the dual short exact sequences (12). By the definitions /page 3 and (1) /

$$(15) \quad \overset{\#}{\Gamma}_{\tau\mathcal{B}}(e_i) = \frac{\partial}{\partial y^i} \quad (i = \overline{1, n})$$

Similarly we have

$$(16) \quad \begin{aligned} \pi\tau\mathcal{B}(\frac{\partial}{\partial x^i}) &= e_i \\ \pi\tau\mathcal{B}(\frac{\partial}{\partial y^i}) &= 0 \end{aligned}$$

since the kernel of $\pi\tau\mathcal{B}$ consists of the fibers of $\nu\mathcal{B}$./page 4 /.

Assume now that $h_{\tau\sigma}(e_i) = S_i^r \frac{\partial}{\partial x^r} - A_i^r \frac{\partial}{\partial y^r}$ with some functions S_i^r and A_i^r . We know that $\rho_{\tau\sigma} \circ h_{\tau\sigma} = id_{(\rho_{\tau\sigma})^* \tau\sigma}$ so

$$e_i = (\rho_{\tau\sigma} \circ h_{\tau\sigma})(e_i) = \rho_{\tau\sigma} \left(S_i^r \frac{\partial}{\partial x^r} - A_i^r \frac{\partial}{\partial y^r} \right) = S_i^r e_r.$$

Therefore $S_i^r \equiv \delta_i^r$. Thus

$$(17) \quad h_{\tau\sigma}(e_i) = \eta_i = \frac{\partial}{\partial x^i} - A_i^j \frac{\partial}{\partial y^j}$$

Now, as we know $\dot{f}_{\tau\sigma} \circ \nu_{\tau\sigma} = id_{\tau\sigma} \circ \nu_{\tau\sigma} - h_{\tau\sigma} \circ \rho_{\tau\sigma}$, and

$$J_m(id_{\tau\sigma} \circ \nu_{\tau\sigma} - h_{\tau\sigma} \circ \rho_{\tau\sigma}) = J_m(\dot{f}_{\tau\sigma} \circ \nu_{\tau\sigma}) = \nu_{\tau\sigma}$$

As we have seen, $\dot{f}_{\tau\sigma}^{-1} \circ \nu_{\tau\sigma} = j_{\tau\sigma}^{-1} = \tau_{\tau\sigma}$ and so

$$\nu_{\tau\sigma} = j_{\tau\sigma}^{-1} \circ (id_{\tau\sigma} \circ \nu_{\tau\sigma} - h_{\tau\sigma} \circ \rho_{\tau\sigma}) = \tau_{\tau\sigma} \circ (id_{\tau\sigma} \circ \nu_{\tau\sigma} - h_{\tau\sigma} \circ \rho_{\tau\sigma}).$$

Finally, in view of (16) and (17) we obtain

$$\nu_{\tau\sigma} \left(\frac{\partial}{\partial x^i} \right) = \tau_{\tau\sigma} \circ (id_{\tau\sigma} \circ \nu_{\tau\sigma} - h_{\tau\sigma} \circ \rho_{\tau\sigma}) \left(\frac{\partial}{\partial x^i} \right) = A_i^j e_j$$

$$\nu_{\tau\sigma} \left(\frac{\partial}{\partial y^i} \right) = e_i$$

Let the covariant derivative ∇_F associated to Γ_F be locally:

$$(18) \quad \nabla_F(e_i) = \Gamma_i^k d z^\alpha e_k$$

Now, according to (6), our constructions of $\overset{*}{\Gamma}$, and (9) we obtain

$$\begin{aligned} \overset{*}{\nabla} \frac{\partial}{\partial x^i} &= K_{\overset{*}{F}} \circ d \left(\frac{\partial}{\partial x^i} \right) = (\dot{f}_{\tau\sigma} \circ K_{\overset{*}{F}} \circ d \nu_{\tau\sigma} + h_{\tau\sigma} \circ K_{\overset{*}{F}} \circ d \rho_{\tau\sigma}) \left(\frac{\partial}{\partial x^i} \right) = \\ &= (\dot{f}_{\tau\sigma} \circ \nabla_F \circ \nu_{\tau\sigma} + h_{\tau\sigma} \circ \nabla_F \circ \rho_{\tau\sigma}) \left(\frac{\partial}{\partial x^i} \right) \end{aligned}$$

Using (16), (17), (18) and (15) we get

$$(19) \quad \overset{*}{\nabla} \left(\frac{\partial}{\partial x^i} \right) = (\dot{f}_{\tau\sigma} \circ \nabla_F) (A_i^j e_j) + (h_{\tau\sigma} \circ \nabla_F)(e_i) = \dot{f}_{\tau\sigma} (A_i^j \Gamma_j^t d z^\alpha e_t + \frac{\partial A_i^t}{\partial x^i} d z^\alpha e_t) +$$

$$h_{\tau\sigma} (\Gamma_i^r d z^\alpha e_r) = \Gamma_i^r d z^\alpha \frac{\partial}{\partial x^r} + \left(\frac{\partial A_i^t}{\partial x^r} + A_i^t \Gamma_r^t - \Gamma_i^r A_r^t \right) d z^\alpha \frac{\partial}{\partial y^t}$$

Similarly we have

$$(20) \quad \overset{*}{\nabla} \frac{\partial}{\partial y^i} = (\dot{f}_{\tau\sigma} \circ \nabla_F \circ \nu_{\tau\sigma} + h_{\tau\sigma} \circ \nabla_F \circ \rho_{\tau\sigma}) \left(\frac{\partial}{\partial y^i} \right) = \Gamma_i^r d z^\alpha \frac{\partial}{\partial y^r}$$

Thus $\overset{*}{\nabla}$ is determined by Γ_i^r and A_i^j .

If $\overset{*}{\Gamma}$ is an extension in the considered sense, then $\overset{*}{\Gamma}$ and $\overset{*}{\Gamma}$ must be equal for certain Γ_i^r and A_i^j . Thus from (13), (19) and (20) we obtain

$$\begin{aligned} G_{\alpha i}^r &= G_{\alpha i}^r \\ H_{\alpha i}^k &= \frac{\partial H_{\alpha i}^k}{\partial z^\alpha} + H_{\alpha i}^k G_{\alpha i}^k - G_{\alpha i}^r H_{\alpha i}^k \\ C_{\alpha i}^k &= C_{\alpha i}^k \end{aligned}$$

Denoting in parenthesis the quantities locally determining a connection we get the

Theorem: A vertical invariant connection $\Gamma(G_{\alpha i}^r, H_{\alpha i}^k, C_{\alpha i}^k)$ is locally the extension of a Finsler connection $\Gamma_F(\bar{G}_{\alpha i}^r)$ with the aid of a connection $\bar{\Gamma}(H_{\alpha i}^k)$ iff $G_{\alpha i}^r$ and $C_{\alpha i}^k$ coincide; $\bar{G}_{\alpha i}^r = C_{\alpha i}^k$ and the partial differential equation system

$$(21) \quad H_{\alpha i}^k = \frac{\partial H_{\alpha i}^k}{\partial z^\alpha} + H_{\alpha i}^k C_{\alpha i}^k - C_{\alpha i}^r H_{\alpha i}^k$$

is integrable.

We note that if a connection Γ is an extension of a Γ_F with the aid of $\bar{\Gamma}$, this fact is independent of the local coordinate system used. Thus, if (21) is integrable in a set of local coordinate systems covering \mathcal{U}_B , then also globally is an extension, but the differentiability of $\bar{\Gamma}$ is not yet answered. However if $\bar{\Gamma}$ / i.e. the solution of (21) / is unique, then Γ is a global extension in our sense.

3. The integrability of (21).

In a linear connection on a manifold M

$$\nabla (H_{\beta}^{\alpha} dz^{\beta} \frac{\partial}{\partial y^{\alpha}}) = (\nabla_{\sigma} H_{\beta}^{\alpha}) dz^{\beta} dz^{\sigma} \frac{\partial}{\partial y^{\alpha}}$$

for any tensor field H_{β}^{α} . It is well known / (4) pp. 124-127 / that

$$(22) \quad 2 \nabla_{[\mu} \nabla_{\rho]} H_{\beta}^{\alpha} = R_{\mu\rho\sigma}^{\alpha} H_{\beta}^{\sigma} - R_{\mu\rho\beta}^{\sigma} H_{\sigma}^{\alpha} - 2 S_{\mu\rho}^{\sigma} \nabla_{\sigma} H_{\beta}^{\alpha}$$

where R and S are the curvature and torsion tensor of the connection.

Let us define the following connection:

$$(23) \quad \nabla_{\alpha}^{\sigma} \frac{\partial}{\partial x^i} = 0, \quad \nabla_{\alpha}^{\sigma} \frac{\partial}{\partial y^i} = \nabla_{\alpha}^{\sigma} \frac{\partial}{\partial y^i}$$

Finally let B_{β}^{α} be given by the definition

$$(24) \quad B_{\beta}^{\alpha} = \begin{cases} H_{\beta}^{\alpha}, & \text{if } \alpha \leq n \text{ and } \beta \leq n \\ 0, & \text{in other cases.} \end{cases}$$

Now in view of (24), (23) and (13) the studied partial equation system (21) can be written in the form

$$(25) \quad (\nabla_{\alpha}^{\sigma})_{\alpha} B_{\beta}^{\sigma} = H_{\alpha\beta}^{\sigma}$$

where $H_{\alpha i}^{\dot{i}}$ are the coefficients appearing in our theorem, while the other coefficients of $H_{\alpha\beta}^{\dot{T}}$ are zeros.

The integrability condition of (25) is

$$\partial_{[\Gamma\delta} (\nabla_{\mu]}^{\circ} B_{\beta}^{\dot{T}} = \partial_{[\Gamma\delta} H_{\mu\beta}^{\dot{T}}$$

This is equivalent with the condition

$$(\nabla_{\Gamma}^{\circ})_{[\delta} (\nabla_{\mu]}^{\circ} B_{\beta}^{\dot{T}} = (\nabla_{\Gamma}^{\circ})_{[\delta} H_{\mu\beta}^{\dot{T}}$$

So, by (22) we get the condition

$$(26) \quad R_{\delta\alpha\mu}^{\dot{T}} B_{\beta}^{\dot{T}} - R_{\delta\alpha\beta}^{\dot{T}} B_{\mu}^{\dot{T}} = 2 (\nabla_{\Gamma}^{\circ})_{[\delta} H_{\mu\beta}^{\dot{T}} + 2 S_{\delta\alpha}^{\mu} H_{\mu\beta}^{\dot{T}}$$

for the integrability of (21). (26) is an ordinary equation system at every x^{α} for the unknowns $B_{\beta}^{\dot{T}}$. If (26) has a solution for $B_{\beta}^{\dot{T}}$ satisfying (24), then our system (21) is also integrable.

REFERENCES

- [1] Duc T.V. "Sur la geometrie differentielle des fibres vectoriels" Kodai Math. Sem. Rep. 26 /1975/, 349-408.
- [2] Greub W., Halperin S., Vanstone R. "Connections, Curvature, and Cohomology", Vol. I-II, Academic Press, New York and London, 1972.
- [3] Oproiu V. "Some properties of the tangent bundle related to the Finsler geometry", The Proceedings of the National Seminar on Finsler Bundles, Brasov, February 1980, p. 195-207.
- [4] Norden A.P. "Affinely connected spaces", Izd. Nauka, Moscow, 1976. /in Russian/.

Lajos Tamássy and Béla Kis
 Institute of Mathematics Debrecen University,
 4010 Debrecen, Pf. 12. Hungary