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NOTE ON STIEFEL-WHITNEY CLASSES OF FLAG MANIFOLDS

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The Stiefel-Whitney characteristic classes seem to contain quite interesting information on real flag manifolds (cf. e.g. [3], [4], [6]).

Let $G(k_1,\ldots,k_r)$ denote the real flag manifold $O(k_1+\ldots+k_r)/O(k_1)\times\ldots\times O(k_r)$, where $k_1,\ldots,k_r (r\geq 2)$ are fixed positive integers. For instance, $G(k_1,k_2)$ is the Grassmann manifold of k_1-planes (or k_2-planes) in real Euclidean k_1+k_2-space.

Recall (cf. [5] for details) that over the manifold $G(k_1,\ldots,k_r)$ one has naturally defined k_i-dimensional vector bundles γ_i $(i=1,\ldots,r)$ with their Whitney sum being trivial bundle. For the tangent bundle one has

(1) $$TG(k_1,\ldots,k_r) = \bigoplus_{1\leq i<j\leq r} \gamma_i \otimes \gamma_j .$$

Moreover, by [1], the Z_2-cohomology algebra $H^*(G(k_1,\ldots,k_r);Z_2)$ can be identified with

$$Z_2[w_1(\gamma_1),\ldots,w_{k_1}(\gamma_1),\ldots,w_1(\gamma_r),\ldots,w_{k_r}(\gamma_r)]/J,$$

where J is an ideal determined by single relation $\prod_{i=1}^{r} w(\gamma_i)=1$. Here $w(\xi) = 1 + w_1(\xi) + w_2(\xi) + \ldots$ means the total Stiefel-Whitney class of a vector bundle ξ. If M is a smooth closed manifold, one puts as usual $w(M)=w(TM)$.

The main purpose of this short note is to illustrate our introductory observation anew by the following

THEOREM. If $r\geq 3$, $k_1\equiv k_2\equiv\ldots\equiv k_r \pmod 2$ and $k_1k_2\ldots k_r>1$, then $w_3(G(k_1,\ldots,k_r))\in H^*(G(k_1,\ldots,k_r);Z_2)$ does not vanish.

As an application, one gets

COROLLARY. If $r\geq 3$, then the flag manifold $G(k_1,\ldots,k_r)$ admits an almost complex structure if and only if $k_1=k_2=\ldots=k_r=1$ and $\dim(G(\underbrace{1,\ldots,1}_{r})) = \binom{r}{2}$ is an even number.

This paper is in final form and no version of it will be submitted for publication elsewhere.

Namely, it is easily verified that the manifold $G(\underbrace{1,\dots,1}_{r})$ is parallelizable.

Therefore, if its dimension is even, this manifold obviously admits an almost complex structure.

Moreover, in order that a real smooth closed manifold M be almost complex, it is necessary that M be even-dimensional, orientable and also that all the integral Stiefel-Whitney classes $W_{2i-1}(M) \in H^{2i-1}(M;Z)$ be zeros (cf. [7, 41.9]), hence the same be true for $w_{2i-1}(M) \in H^{2i-1}(M;Z_2)$.

Keeping in mind that $k_1 \equiv k_2 \equiv \dots \equiv k_r \pmod 2$ is equivalent to orientability of $G(k_1,\dots,k_r)$ (cf. [3]), we get Corollary as a consequence of Theorem indeed.

Proof of Theorem. Without loss of generality, we suppose $k_1 \leqslant k_2 \leqslant \dots \leqslant k_r$. Hence $k_1 k_2 \dots k_r > 1$ implies clearly $k_r \geqslant 2$.

Consider first the case r=3. If $k_1 \equiv k_2 \equiv k_3 \pmod 2$, we compute from (1) (cf. [3] if needed)

$$w_2(G(k_1,k_2,k_3)) = \left[1 + \binom{k_1}{2} + \binom{k_3}{2}\right] w_1^2(\gamma_1) + \left[1 + \binom{k_2}{2} + \binom{k_3}{2}\right] w_1^2(\gamma_2) +$$

$$+ w_1(\gamma_1) w_1(\gamma_2).$$

Since $w_1(G(k_1,k_2,k_3))$ is now zero, the Wu formula yields

$$w_3(G(k_1,k_2,k_3)) = w_1^2(\gamma_1) w_1(\gamma_2) + w_1(\gamma_1) w_1^2(\gamma_2).$$

By direct finding a basis in $H^*(G(k_1,k_2,k_3);Z_2)$ or by applying the Leray-Hirsch Theorem to the obvious differentiable fibre bundle

$$\begin{array}{ccc} G(k_2,k_3) & \hookrightarrow & G(k_1,k_2,k_3) \\ & & \downarrow \\ & & G(k_1,k_2+k_3) \end{array}$$

one proves the assertion.

Now recall ([2]) that when $\begin{array}{ccc} F & \overset{i}{\hookrightarrow} & E \\ & & \downarrow p \\ & & B \end{array}$ is a differentiable fibre bundle, then one has $TE = p^*(TB) \oplus \eta$, where η is the "tangent bundle along the fibres". So, if F is connected, $w_j(F) \neq 0$ implies $w_j(E) \neq 0$.

This, when applied to the fibre bundle

$$\begin{array}{ccc} G(k_2,\dots,k_r) & \hookrightarrow & G(k_1,\dots,k_r) \\ & & \downarrow \\ & & G(k_1,k_2+\dots+k_r), \end{array}$$

with an obvious induction, proves Theorem completely.

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