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In: Jarolím Bureš and Vladimír Souček (eds.): Proceedings of the Winter School "Geometry and Physics". Circolo Matematico di Palermo, Palermo, 1990. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 22. pp. [177]–191.

Persistent URL: `http://dml.cz/dmlcz/701471`

NATURAL TRANSFORMATIONS OF WEIL FUNCTORS INTO BUNDLE FUNCTORS

Włodzimierz M. Mikulski

Abstract. We deduce that the set of all natural transformations of the Weil functor T^A of A-velocities into a bundle functor F is bijectively related to the set

$$\{v \in F_0\mathbb{R}^k : \forall f \in C^\infty(\mathbb{R}^k,\mathbb{R}^{k+1})\ (j^Af=j^Ai_k \Longrightarrow Ff(v)=Fi_k(v)\ )\},$$

provided A is a Weil algebra in k variables and where $i_k:\mathbb{R}^k \longrightarrow \mathbb{R}^{k+1}$ is given by $i_k(x)=(x,0)$. In the case where F is a linear bundle functor we deduce that the dimension of the vector space of all natural transformations of T^A into F is finite and is less than or equal to $\dim(F_0\mathbb{R}^k)$. We construct a linear bundle functor G such that the vector space of all natural transformations of G into G is infinite dimensional. We determine the spaces of all natural transformations of Weil functors into linear functors of higher order tangent bundles. Corollary 4.2 shows that any bundle functor has (locally) a finite order .

1. Bundle functors. Throughout the paper all manifolds are assumed to be paracompact, without boundary, second countable, finite dimensional and smooth, i.e of class C^∞. In general maps will be assumed to be C^∞, unless the smoothness should be proved.

Let Mf be the category of all manifolds and all maps, FM be the category of all fibered manifolds and their morphisms and $B: \mathcal{FM} \longrightarrow \mathcal{Mf}$ be the base functor. Given a functor $F: \mathcal{Mf} \longrightarrow \mathcal{FM}$ satisfying $B\circ F= \mathrm{id}_{\mathcal{Mf}}$, we denote by $p^F_M: FM \longrightarrow M$

"This paper is in final form and no version of it will be submited for publication elsewhere."

its value on a manifold M and by $F_x f: F_x M \longrightarrow F_{f(x)} N$ the restriction of its value $Ff: FM \longrightarrow FN$ in $f: M \longrightarrow N$ to the fibres of FM over x and of FN over f(x), $x \in M$.

Definition 1.1 ([8]) A bundle functor on $\underline{Mf}$ is a functor $F: \underline{Mf} \longrightarrow \underline{FM}$ satisfying $B \circ F = id_{\underline{Mf}}$ and the localization condition: if $i: U \longrightarrow M$ is the inclusion of an open subset, then $Fi: FU \longrightarrow (p^F_M)^{-1}(U)$ is a diffeomorphism.

Let M,N,P be manifold. A parametrized system of smooth maps $f_p: M \longrightarrow N$, $p \in P$ is said to be smoothly parametrized, if the resulting map $f: M \times P \longrightarrow N$ is of class C^∞.

Proposition 1.1 ([8]) Every bundle functor $F: \underline{Mf} \longrightarrow \underline{FM}$ satisfies the regularity condition: if $f: M \times P \longrightarrow N$ is a smoothly parametrized family, then the family $\widetilde{Ff}: FM \times P \longrightarrow FN$ defined by $(\widetilde{Ff})_p = F(f_p)$ is also smoothly parametrized.

We will cite the proof of the proposition in Section 9.

2. Weil functors. Let E(k), $k \in \mathbb{N}$ be the algebra of all germs at zero of smooth functions on $\mathbb{R}^k$ into $\mathbb{R}$, $\underline{m}(k)$ the ideal of all germs from E(k) vanishing at zero and $\underline{m}(k)^{r+1}$ its (r+1) power. Any ideal A in E(k) satisfying the condition $\underline{m}(k) \supset A \supset \underline{m}(k)^{r+1}$ (for some integer $r \geqslant 0$) will be called a Weil ideal and the corresponding Weil algebra in k variables is defined to be the factor algebra $A = E(k)/A$.

Let M be a manifold and $A = E(k)/A$ be a Weil algebra. Let E(M,x) be the set of all germs at a point $x \in M$ of smooth functions on M into $\mathbb{R}$. We recall the following definition.

Definition 2.1 ([5]) Two maps $g,h: \mathbb{R}^k \longrightarrow M$, $g(0)=h(0) = x$, are said to be A-equivalent, if $\varphi \circ g - \varphi \circ h \in A$ for every germ $\varphi \in E(M,x)$. Such an equivalence class will be denoted by $j^A g$ and called an A-velocity on M. The point g(0) will be said to be the target of $j^A g$.

Denote by $T^A M$ the set of all A-velocities on M. The target map is the projection $p_M: T^A M \longrightarrow M$. Every chart (U,φ), $\varphi = (\varphi^1, \dots, \varphi^n)$ on M determines a chart $((p_M)^{-1}(U), \tilde{\varphi})$ on $T^A M$ in the following way:

$$\tilde{\varphi}(j^A g) = (j^A(\varphi^1 \circ g), \dots, j^A(\varphi^n \circ g)) \in A \times \dots \times A \simeq \mathbb{R}^{n(\dim A)}$$

Hence $T^A M$ is an $(n\dim A)$-dimensional manifold. Further, for every $f:M \longrightarrow N$ we define $T^A f: T^A M \longrightarrow T^A N$ by $T^A f(j^A g) = j^A(f\circ g)$. Obviously, T^A is a bundle functor. We call T^A a Weil functor of A-velocities. The functor was described by A. Morimoto [11] as another description of a Weil functor of near A-points [15]. For $A = \underline{m}(k)^{r+1}$ such a functor coincides with the k^r-velocities functor studied by C. Ehresmann [2]. The k^r-velocities functor maps a manifold M to the bundle $T^{r,k}M = J^r_0(\mathbb{R}^k, M)$ of all r-jets at zero of maps of $\mathbb{R}^k$ into M and a map $f:M \longrightarrow N$ to the extension $T^{r,k}f : T^{r,k}M \longrightarrow T^{r,k}N$ defined by the composition of jets.

3. An order theorem. The crucial point in our studies is the following order theorem. From now on i_k will denote the map $i_k:\mathbb{R}^k \longrightarrow \mathbb{R}^{k+1}$ given by $i_k(x) = (x,0)$.

Theorem 3.1 Let F be a bundle functor, k a natural number, $A = E(k)/A$ a Weil algebra and $v \in F_0\mathbb{R}^k$ a point. Suppose that $j^A\varphi = j^A i_k$ implies $F\varphi(v) = Fi_k(v)$ for any map $\varphi:\mathbb{R}^k \longrightarrow \mathbb{R}^{k+1}$. Then for any two maps $f,g:\mathbb{R}^k \longrightarrow M$ with $j^A f = j^A g$ we have $Ff(v) = Fg(v)$.

Proof. Let F, k, A and v satisfy the assumptions of the theorem. We shall prove the following lemmas.

Lemma 3.1 If $f:\mathbb{R}^k \longrightarrow \mathbb{R}^k$ is a map such that $j^A f = j^A id$, then $Ff(v) = v$. (We denote by id the identity map on $\mathbb{R}^k$.)

Proof of Lemma 3.1. Let $p_k:\mathbb{R}^{k+1} = \mathbb{R}^k \times \mathbb{R} \longrightarrow \mathbb{R}^k$ be the canonical projection. Since $j^A(i_k\circ f) = j^A(i_k)$, we have that $Fi_k(v) = F(i_k\circ f)(v)$. Therefore $Ff(v) = F(p_k\circ i_k\circ f)(v) = Fp_k\circ F(i_k\circ f)(v) = Fp_k\circ Fi_k(v) = Fid(v) = v$. ∎

Lemma 3.2 Suppose $f,g: (\mathbb{R}^k,0) \longrightarrow (\mathbb{R}^k,0)$ are maps such that $Jac_0(g) \neq 0$ and $j^A f = j^A g$. Then $Ff(v) = Fg(v)$.

Proof of Lemma 3.2. Let $h:(\mathbb{R}^k,0) \longrightarrow (\mathbb{R}^k,0)$ be a map such that $germ_0(g\circ h) = germ_0(h\circ g) = germ_0(id)$. Of course, $j^A(h\circ f) = j^A(id)$. Therefore, by Lemma 3.1 and the localization condition, we get $Ff(v) = F(g\circ h)\circ Ff(v) = Fg\circ F(h\circ f)(v) = Fg(v)$. ∎

Lemma 3.3 If $f,g:(\mathbb{R}^k,0) \longrightarrow (\mathbb{R}^k,0)$ are maps such that

$j^A f = j^A g$, then $Ff(v) = Fg(v)$.

Proof of Lemma 3.3. Consider one parameter families $f_t = f + t\,id$, $g_t = g + t\,id$, $t \in \mathbb{R}$. Since their Jacobians at 0 are certains non-zero polynomials in t , f_t and g_t are local diffeomorphisms in neighbourhoods of 0 except a finite number values of t . Since $j^A f_t = j^A g_t$ for all t , Lemma 3.2 implies $Ff_t(v) = Fg_t(v)$ except a finite number values of t . Then the regularity condition (Proposition 1.1) yields $Ff_0(v) = Fg_0(v)$. ■

Lemma 3.4. Let $f,g : (\mathbb{R}^k,0) \longrightarrow (\mathbb{R}^m,0)$ be maps such that $j^A f = j^A g$ and $m < k$. Then $Ff(v) = Fg(v)$.

Proof of Lemma 3.4. Define $j: \mathbb{R}^m \longrightarrow \mathbb{R}^k$ by $j(y) = (y,0)$, $0 \in \mathbb{R}^{k-m}$ and $p: \mathbb{R}^k = \mathbb{R}^m \times \mathbb{R}^{k-m} \longrightarrow \mathbb{R}^m$ to be the obvious projection. Since $j^A(j\circ f) = j^A(j\circ g)$, Lemma 3.3 implies $F(j\circ f)(v) = F(j\circ g)(v)$. Hence $Ff(v) = F(p\circ j\circ f)(v) = Fp\circ F(j\circ f)(v) = Fp\circ F(j\circ g)(v) = Fg(v)$. ■

Lemma 3.5 For every functions $h^1,\dots,h^m : \mathbb{R}^k \longrightarrow \mathbb{R}$ $(m \geq k+2)$ such that $j^A h^1 = \dots = j^A h^m = j^A 0$, we have $F(id+(h^1,\dots,h^k),h^{k+1},\dots,h^m)(v) = F(id+(h^1,\dots,h^k),0,h^{k+2},\dots,h^m)(v)$.

Proof of Lemma 3.5. Put $h=(h^1,\dots,h^k)$. Define $H: \mathbb{R}^{k+1} \longrightarrow \mathbb{R}^m$ by $H(x,y)=(x+h(x),y,h^{k+2}(x),\dots,h^m(x))$, where $x \in \mathbb{R}^k$ and $y \in \mathbb{R}$. It is obvious that $H\circ(id,h^{k+1}) = (id+h,h^{k+1},\dots,h^m)$ and $H\circ i_k = (id+h,0,h^{k+2},\dots,h^m)$. By using the equality $j^A(id,h^{k+1}) = j^A i_k$, we get $F(id,h^{k+1})(v) = Fi_k(v)$. Therefore $F(id+h,h^{k+1},\dots,h^m)(v) = F(H\circ(id,h^{k+1}))(v) = FH\circ F(id,h^{k+1})(v) = FH\circ Fi_k(v) = F(id+h,0,h^{k+2},\dots,h^m)(v)$. ■

Lemma 3.6 If $h^1,\dots,h^m : \mathbb{R}^k \longrightarrow \mathbb{R}$ $(m \geq k+2)$ are functions such that $j^A h^1 = \dots = j^A h^m = j^A 0$, then $F(id+h,h^{k+1},\dots,h^m)(v) = F(id+h,0,\dots,0)(v)$, where $h=(h^1,\dots,h^k)$.

Proof of Lemma 3.6. By using the induction on s we shall prove that $F(id+h,h^{k+1},\dots,h^m)(v) = F(id+h,0,\dots,0,h^{k+s+1},\dots,h^m)(v)$.

If $s=1$, then the assertion is given in Lemma 3.5. Assume that the assertion is proved for $s=s^*$. Suppose $k+s^*+1 \leq m$. Let ρ be the transposition exchanging $k+s^*+1$ and $k+1$ in the sequence $(1,\dots,m)$. Define $S: \mathbb{R}^m \longrightarrow \mathbb{R}^m$ by

$S(y^1,\dots,y^m)=(y^{\varrho(1)},\dots,y^{\varrho(m)})$. By Lemma 3.5 with $h^{k+s^*+1},0,\dots,0,h^{k+s^*+2},\dots,h^m$ playing the role of $h^{k+1},\dots,h^m$ we have $F(S\circ(id+h,0,\dots,0,h^{k+s^*+1},\dots,h^m))(v) = F(id+h,0,\dots,0,h^{k+s^*+2},\dots,h^m)(v)$. Hence $F(id+h,h^{k+1},\dots,h^m)(v) = FS^{-1}\circ F(S\circ(id+h,0,\dots,0,h^{k+s^*+1},\dots,h^m))(v) = F(S^{-1}\circ(id+h,0,\dots,0,h^{k+s^*+2},\dots,h^m))(v) = F(id+h,0,\dots,0,h^{k+s^*+2},\dots,h^m)(v)$ as required. ■

<u>Lemma 3.7</u> Let $i^m:\mathbb{R}^k\longrightarrow\mathbb{R}^m$ $(m\geqslant k+1)$ be given by $i^m(x) = (x,0)$, $0\in\mathbb{R}^{m-k}$. Suppose that $f:\mathbb{R}^k\longrightarrow\mathbb{R}^m$ is a function such that $j^Af=j^Ai^m$. Then $Ff(v)=Fi^m(v)$.

<u>Proof of Lemma 3.7</u>. If $m=k+1$, then $i^m=i_k$ and therefore $Ff(v)=Fi^m(v)$. So, we assume that $m\geqslant k+2$. We can choose functions $h^1,\dots,h^m:\mathbb{R}^k\longrightarrow\mathbb{R}$ such that $j^Ah^1=\dots=j^Ah^m=j^A0$ and $f=(id+h,h^{k+1},\dots,h^m)$, where $h=(h^1,\dots,h^k)$. By Lemma 3.6 we have $Ff(v)=F(id+h,0,\dots,0)(v)$. Since $j^A(id+h)=j^Aid$, Lemma 3.1 implies $F(id+h)(v)=v$. It is easily seen that $(id+h,0,\dots,0) =i^m\circ(id+h)$. Therefore $Ff(v) = F(id+h,0,\dots,0)(v)= F(i^m\circ(id+h))(v) = Fi^m\circ F(id+h)(v) = Fi^m(v)$. ■

<u>Lemma 3.8</u> If $f,g:(\mathbb{R}^k,0)\longrightarrow(\mathbb{R}^m,0)$ $(m\geqslant k+1)$ are two maps such that $\mathrm{rank}_0 f = \mathrm{rank}_0 g = k$ and $j^Af = j^Ag$, then $Ff(v) = Fg(v)$.

<u>Proof of Lemma 3.8.</u> By the rank theorem there exist two diffeomorphisms $\psi_i:(V_i,0)\longrightarrow(W_i,0)$, $i=1,2$, $V_1,W_1\in\mathrm{top}\mathbb{R}^k$, $V_2,W_2\in\mathrm{top}\mathbb{R}^m$, such that $\psi_2\circ g\circ\psi_1 = i^m$ on some open neighbourhood of $0\in\mathbb{R}^k$. (We recall that $i^m:\mathbb{R}^k\longrightarrow\mathbb{R}^m$ is given by $i^m(x)=(x,0)$.) Let id_{m-k} be the identity map on $\mathbb{R}^{m-k}$. By $i^m\circ\psi_1^{-1} = (\psi_1^{-1}\times id_{m-k})\circ i^m$, we have that $(\psi_1\times id_{m-k})\circ\psi_2\circ g = i^m$ on some open neighbourhood of $0\in\mathbb{R}^k$. Let $\tilde f:\mathbb{R}^k\longrightarrow\mathbb{R}^m$ be a function of class C^∞ such that $\mathrm{germ}_0\tilde f = \mathrm{germ}_0((\psi_1\times id_{m-k})\circ\psi_2\circ f)$ and $\tilde\psi:\mathbb{R}^m\longrightarrow\mathbb{R}^m$ a function of class C^∞ such that $\mathrm{germ}_0(\psi_2^{-1}\circ(\psi_1^{-1}\times id_{m-k})) = \mathrm{germ}_0\tilde\psi$. Since $j^A\tilde f = j^Ai^m$, Lemma 3.7 implies that $F\tilde f(v) = Fi^m(v)$. But $\mathrm{germ}_0(\tilde\psi\circ\tilde f)= \mathrm{germ}_0 f$ and $\mathrm{germ}_0(\tilde\psi\circ i^m) = \mathrm{germ}_0 g$. Therefore, by the localization condition, we have $Ff(v) = F(\tilde\psi\circ\tilde f)(v) = F\tilde\psi\circ F\tilde f(v) = F\tilde\psi\circ Fi^m(v) = F(\tilde\psi\circ i^m)(v) = Fg(v)$. ■

<u>Lemma 3.9</u> Let $f,g:(\mathbb{R}^k,0)\longrightarrow(\mathbb{R}^m,0)$ $(m\geqslant k+1)$ be two maps such that $j^Af = j^Ag$. Then $Ff(v) = Fg(v)$.

<u>Proof of Lemma 3.9.</u> Consider one-parameter families $f_t =$

$= f + t\cdot i^m$, $g_t = g + t\cdot i^m$, $t \in \mathbb{R}$. Define $p:\mathbb{R}^m = \mathbb{R}^k \times \mathbb{R}^{m-k} \longrightarrow \mathbb{R}^k$ to be the projection. Since $p\circ f_t = p\circ f + t\cdot id$ and $p\circ g_t = p\circ g + t\cdot id$, so by using similar arguments as in the proof of Lemma 3.3 , we obtain that $p\circ f_t$ and $p\circ g_t$ are local diffeomorphisms in neighbourhoods of $0 \in \mathbb{R}^k$ except a finite number values of t. Therefore $\mathrm{rank}_0 f_t = \mathrm{rank}_0 g_t = k$ except a finite number values of t. Since $j^A f_t = j^A g_t$ for all t , Lemma 3.8 implies $Ff_t(v) = Fg_t(v)$ except a finite number values of t. Then the regularity condition (Proposition 1.1) yields $Ff_0(v) = Fg_0(v)$. ∎

We are now in position to prove Theorem 3.1. Consider arbitrary functions $f,g:\mathbb{R}^k \longrightarrow M$ such that $j^A f = j^A g$. Choose a chart (U,φ) on M satisfying $\varphi(U) = \mathbb{R}^{\dim M}$ and $\varphi(f(0)) = 0$. Let $\tilde{f}, \tilde{g}:(\mathbb{R}^k,0) \longrightarrow (\mathbb{R}^{\dim M},0)$ be two functions of class C^∞ such that $\mathrm{germ}_0 f = \mathrm{germ}_0(\varphi^{-1}\circ\tilde{f})$ and $\mathrm{germ}_0 g = \mathrm{germ}_0(\varphi^{-1}\circ\tilde{g})$. Since $j^A\tilde{f} = j^A\tilde{g}$, Lemma 3.3, Lemma 3.4 and Lemma 3.9 yield $F\tilde{f}(v) = F\tilde{g}(v)$. Hence, by the localization condition, we get $Ff(v) = F(\varphi^{-1}\circ\tilde{f})(v) = F\varphi^{-1}\circ F\tilde{f}(v) = F\varphi^{-1}\circ F\tilde{g}(v) = F(\varphi^{-1}\circ\tilde{g})(v) = Fg(v)$. Theorem 3.1 is proved. ∎

4. Corollaries. From Theorem 3.1 we get the following corollary.

Corollary 4.1 Let $F: \underline{Mf} \longrightarrow \underline{FM}$ be a bundle functor, $r \geq 0$ an integer, k a natural number and $v \in F_0\mathbb{R}^k$ a point. Suppose that $j_0^r\varphi = j_0^r i_k$ implies $F\varphi(v) = Fi_k(v)$ for any map $\varphi:\mathbb{R}^k \longrightarrow \mathbb{R}^{k+1}$. Then for any maps $f,g: \mathbb{R}^k \longrightarrow M$ with $j_0^r f = j_0^r g$ we have $Ff(v) = Fg(v)$.

Proof. We apply Theorem 3.1 in the case where $A = \underline{m}(k)^{r+1}$. ∎

Let $F:\underline{Mf} \longrightarrow \underline{FM}$ be a bundle functor on $\underline{Mf}$. If we replace the category $\underline{Mf}$ by the category $\underline{Mf_m}$ of all m-dimensional manifolds and their local diffeomorphisms, we obtain the classical concept of a natural bundle in dimension m introduced by Nijenhuis ,[12], and Palais- Terng ,[13]. Hence the restriction F_m of F to $\underline{Mf_m}$ is a natural bundle in dimension m . According to Palais-Terng ,[13], every natural bundle has a finite order . Let F_m has a order $r(m)$. We recall that $r(m) := \min\{r \in \mathbb{N}\cup\{\infty\}: \ j_x^r f = j_x^r g$ implies $F_x f = F_x g$

for any two local diffeomorphisms f,g of m-dimensional manifolds and any $x \in \mathrm{dom}(f) \cap \mathrm{dom}(g) \}$. (In [3], [13] and [16] estimates of $r(m)$ are given .)

I. Kolář and J. Slovák proved in [8] the following result.

Proposition 4.1 Let F be a bundle functor , $M,N \in \underline{Mf}$. Write $m=\dim M$, $n=\dim N$ and $r(m,n) = r(\max(m,n))$. Then for any maps $f,g:M \longrightarrow N$, $j_x^{r(m,n)}f = j_x^{r(m,n)}g$ implies $F_x f = F_x g$.

On the other hand we constructed in [10] a bundle functor of infinite order , i.e with an unbounded sequence of $r(m)$. Therefore the following corollary is interesting.

Corollary 4.2 Every bundle functor F has locally a finite order. More precisely, for any maps $f,g: M \longrightarrow N$, $j_x^{r(\dim M+1)}f = j_x^{r(\dim M+1)}g$ implies $F_x f = F_x g$.

Proof. Consider two maps $f,g:M \longrightarrow N$ such that $j_x^{r(m+1)}f = j_x^{r(m+1)}g$, where $x \in M$ and $m=\dim M$. By using a chart around x , we can assume that $M=\mathbb{R}^m$ and $x=0$. By Proposition 4.1 we get $j_0^{r(m+1)}\varphi = j_0^{r(m+1)}i_m$ implies $F_0\varphi = F_0 i_m$ for any map $\varphi:\mathbb{R}^m \longrightarrow \mathbb{R}^{m+1}$. (An independent proof of the last fact is the following: Define $\Phi:\mathbb{R}^{m+1} \longrightarrow \mathbb{R}^{m+1}$ by $\Phi(x,y)=\varphi(x)+(0,y)$, $x \in \mathbb{R}^m$, $y \in \mathbb{R}$. Recall that $i_m: \mathbb{R}^m \longrightarrow \mathbb{R}^{m+1}$ is given by $i_m(x) = (x,0)$. Since $j_0^{r(m+1)}\varphi = j_0^{r(m+1)}i_m$, we have that $j_0^{r(m+1)}\Phi = j_0^{r(m+1)}\mathrm{id}$. Therefore $F_0\Phi = F_0\mathrm{id}$. But $\Phi \circ i_m = \varphi$. Hence $F_0\varphi = F_0(\Phi \circ i_m) = F_0\Phi \circ F_0 i_m = F_0 i_m$.) Therefore , by Corollary 4.1 with $r=r(m+1)$ and $k=m$, we obtain that $F_x f = F_x g$. This completes the proof of the corollary. ∎

An unsolved problem. According to Corollary 4.1 we have the following unsolved problem. Let F be a bundle functor such that F_m has order $r(m)$. For each natural number m , find the minimal number $R(m)$ such that for any maps $f,g:M \longrightarrow N$, $m=\dim M$, $x \in M$, $j_x^{R(m)}f = j_x^{R(m)}g$ implies $F_x f = F_x g$. From Corollary 4.2 it follows that $R(m) \leq r(m+1)$. On the other hand it is obvious that $R(m) \geq r(m)$. Is $R(m)$ equal to $r(m)$?

5. Natural transformations of Weil functors into bundle functors. We recall the following definition.

Definition 5.1 Let F and G be two bundle functors on $\underline{Mf}$. A family of C^∞ maps $I(M):FM \longrightarrow GM$, $M \in \underline{Mf}$ is called a natural transformation of F into G if for any $f:M \longrightarrow N$ $I(N)\circ Ff = Gf\circ I(M)$.

Remark. One can show that for every natural transformation $I:F \longrightarrow G$ and $M \in \underline{Mf}$ $p_M^G \circ I(M) = p_M^F$. A simple proof of this fact is given in [7].

From now on $Trans(F,G)$ will denote the set of all natural transformations of F into G. (This is a set because any natural transformation $I:F \longrightarrow G$ is uniquely determined by the sequence $I(\mathbb{R}^m)$, $m=0,1,2,\dots$.) If $A = E(k)/A$ is a Weil algebra and F a bundle functor, then define $Adm(A,F)$ to be the set

$$\{v \in F_0\mathbb{R}^k : \forall f \in C^\infty(\mathbb{R}^k,\mathbb{R}^{k+1}) \ (j^A f = j^A i_k \Longrightarrow Ff(v) = Fi_k(v))\} ,$$

where $i_k:\mathbb{R}^k \longrightarrow \mathbb{R}^{k+1}$ is given by $i_k(x) = (x,0)$. We prove the following theorem.

Theorem 5.1 Let $F:\underline{Mf} \longrightarrow \underline{FM}$ be a bundle functor and $A=E(k)/A$ a Weil algebra. Then the function $J:Trans(T^A,F) \longrightarrow Adm(A,F)$ given by $J(I)=I(\mathbb{R}^k)(j^A(id_k))$ (where id_k is the identity map on $\mathbb{R}^k$) is a bijection. The inverse bijection is of the form $Adm(A,F) \ni v \longrightarrow I^v \in Trans(T^A,F)$ where $I^v(M):T^AM \longrightarrow FM$ is given by $I^v(M)(j^A f)=Ff(v)$.

Proof. Consider $I \in Trans(T^A,F)$. If $j^A f = j^A i_k$, then $Fi_k(I(\mathbb{R}^k)(j^A(id_k))) = I(\mathbb{R}^{k+1})\circ T^A i_k(j^A(id_k)) = I(\mathbb{R}^{k+1})(j^A i_k) = I(\mathbb{R}^{k+1})(j^A f) = I(\mathbb{R}^{k+1})\circ T^A f(j^A(id_k)) = Ff(I(\mathbb{R}^k)(j^A(id_k)))$. Hence $I(\mathbb{R}^k)(j^A(id_k)) \in Adm(A,F)$. Therefore J is well-defined.

Now, suppose that I', $I'' \in Trans(T^A,F)$ are such that $I'(\mathbb{R}^k)(j^A(id_k)) = I''(\mathbb{R}^k)(j^A(id_k))$. Then $I'(M)(j^A f) = I'(M)\circ T^A f(j^A(id_k)) = Ff\circ I'(\mathbb{R}^k)(j^A(id_k)) = Ff\circ I''(\mathbb{R}^k)(j^A(id_k)) = I''(M)(j^A f)$ for any $j^A f \in T^A M$. Hence J is a injection.

The main difficulty in proving Theorem 5.1 is to show that J is a surjection. Consider $v \in Adm(A,F)$. By Theorem 3.1 the condition $j^A f = j^A g$ implies $Ff(v) = Fg(v)$. Therefore $I^v(M):T^AM \longrightarrow FM$ is well-defined. For any $h:M \longrightarrow N$ and any $j^A f$ we have $I^v(N)\circ T^A h(j^A f) = I^v(N)(j^A(h\circ f)) = F(h\circ f)(v) = Fh\circ Ff(v) = Fh\circ I^v(M)(j^A f)$. It is clear that $I^v(\mathbb{R}^k)(j^A(id_k)) = v$.

Hence the theorem is proved, provided $I^V(M)$ is of class C^∞.

We have to show that $I^V(M)$ is of class C^∞. Since $I^V(M)\circ T^A\varphi^{-1} = F\varphi^{-1}\circ I^V(\mathbb{R}^n)$ for any chart φ on M, it is sufficient to show that $I^V(\mathbb{R}^n)$ is of class C^∞ for every natural number n. We shall use the following lemma, which is a stronger version of Boman's Theorem, [1]:

Lemma 5.1 Let $f:M\longrightarrow N$ be a function of two positive dimensional manifolds such that for every C^∞ function $\gamma:\mathbb{R}\longrightarrow M$ $f\circ\gamma$ is of class C^∞. Then f is of class C^∞.

Proof of the lemma. Recall that in the theorem of Boman M and N are $\mathbb{R}^p$ and $\mathbb{R}^q$ respectively. At first we assume that f is continuous. Consider $x_0\in M$. Choose a chart (U,φ) on N near $f(x_0)$ such that $\varphi(U)=\mathbb{R}^{\dim N}$. There exists a chart (V,ψ) on M near x_0 such that $\psi(V)=\mathbb{R}^{\dim M}$ and $f(V)\subset U$. By Boman's theorem and the assumption of Lemma 5.1 we get that $\varphi\circ f\circ\psi^{-1}$ is of class C^∞. Therefore f is of class C^∞.

Hence we have to show that f is continuous. Suppose that f is discontinuous in $y_0\in M$. Choose a chart $(\tilde V,\tilde\psi)$ on M near y_0 such that $\tilde\psi(\tilde V)=\mathbb{R}^{\dim M}$ and $\tilde\psi(y_0)=0$. By replacing f by $f\circ\tilde\psi^{-1}$ we can assume that $M=\mathbb{R}^m$ and $y_0=0$. There exist a sequence of points $x_i\in\mathbb{R}^m$ $(i=1,2,\dots)$ and a neighbourhood $\tilde U$ of $f(0)$ such that $x_i\longrightarrow 0$ and $f(x_i)\notin\tilde U$ for all i. By passing to subsequences we can assume that $\|x_i\|<\exp(-i)$ for all i. By the Whitney extension theorem [14] there exist a function $\gamma:\mathbb{R}\longrightarrow\mathbb{R}^m$ of class C^∞ such that $\gamma(1/i)=x_i$ for all i. But $f\circ\gamma$ is of class C^∞. Hence $f(x_i)=f\circ\gamma(1/i)\longrightarrow f\circ\gamma(0)=f(0)$. This is a contradiction and the lemma is proved. ■

Now, it is sufficient to show that $I^V(\mathbb{R}^n)\circ\gamma$ is of class C^∞ for any C^∞ curve $\gamma:\mathbb{R}\longrightarrow T^A\mathbb{R}^n$. Suppose that $\underline{m}(k)^{r+1}\subset(A)$. Let $\gamma:\mathbb{R}\longrightarrow T^A\mathbb{R}^n$ be an arbitrary C^∞ curve. There exists a linear section $s:A\longrightarrow E(k)/\underline{m}(k)^{r+1}$ with respect to the linear projection $E(k)/\underline{m}(k)^{r+1}\longrightarrow A$ given by $j_0^rf\longrightarrow j^Af$. Put $\gamma(t)=j^A(f_t^1,\dots,f_t^n)$ and $j_0^r(F_t^i)=s(j^A(f_t^i))$, $i=1,\dots,n$. There exist C^∞ maps $\Phi^i:\mathbb{R}\times\mathbb{R}^k\longrightarrow\mathbb{R}$ such that $j_0^r(F_t^i)=j_0^r(\Phi^i(t,\cdot))$ for $i=1,\dots,n$. For example,

$\Phi^i(t,x) = \sum_{|\alpha|\leq r} (1/\alpha!\ D^\alpha F_t^i(0)x^\alpha)$. It is obvious that $j^A(\Phi_t^1,\dots,\Phi_t^n) = \gamma(t)$, where $\Phi_t^i(x) = \Phi^i(t,x)$. By Proposition 1.1, we have that the mapping $I^V(\mathbb{R}^n)\circ\gamma$ is of class C^∞ because $I^V(\mathbb{R}^n)\circ\gamma(t) = I^V(\mathbb{R}^n)(j^A(\Phi_t^1,\dots,\Phi_t^n)) = F(\Phi_t^1,\dots,\Phi_t^n)(v)$. This finishes the proof of the theorem. ∎

As a special case of Theorem 5.1 ($(A) = \underline{m}(k)^{r+1}$) we have the following corollary.

Corollary 5.1 Let F be a bundle functor on $\underline{Mf}$ such that F_{k+1}, the restriction of F to the subcategory of $(k+1)$-dimensional manifolds and its local diffeomorphisms, has order $r(k+1)$. Suppose that $r \geq r(k+1)$. Then there is a bijection between $\mathrm{Trans}(T^{r,k},F)$ and $F_0\mathbb{R}^k$ given by $I \longmapsto I(\mathbb{R}^k)(j_0^r id_k)$.

6. Natural transformations of Weil functors into Weil functors. Let $A=E(k)/(A)$ and $B=E(p)/(B)$ be two Weil algebras. In [5], I. Kolař introduced the following definition.

Definition 6.1 We say that $j^B f \in T_0^B\mathbb{R}^k$ is an A admisible B velocity if $j^B(\varphi\circ f)=j^B 0$ for all $\varphi\in(A)$.

It is easy to show that the set of all A admisible B velocities is equal to $\mathrm{Adm}(A,T^B)$. Therefore we have the following corollary. (This corollary was deduced by I. Kolař [5].)

Corollary 6.1 There is a bijection between the natural transformations $I:T^A \longrightarrow T^B$ and the A admisible B velocities given by $I(\mathbb{R}^k)(j^A(id_k))$.

7. Natural transformations of Weil functors into linear functors of higher order tangent bundles. A class of well known functors in differential geometry can be constructed as follows, see e.g [4],[6]. Given two integers $q,r \geq 1$ and a manifold M, we put $T_q^{r*}M=J^r(M,\mathbb{R}^q)_0$, the set of all r-jets of M into $\mathbb{R}^q$ with target 0. One can see that $T_q^{r*}M$ is a vector bundle with standard fibre $J_0^r(\mathbb{R}^m,\mathbb{R}^q)_0$, provided $\dim M=m$. Let T_q^rM be the dual vector bundle of $T_q^{r*}M$. Given any r-jet A from $J_x^r(M,N)_y$, the composition of jets determines a linear map from the fibre $(T_q^{r*}N)_y$ over $y\in N$ into the fibre $(T_q^{r*}M)_x$. Hence any smooth map $f:M\longrightarrow N$ induces a linear morphism $T_q^{r*}f:f^!T_q^{r*}N \longrightarrow T_q^{r*}M$, where $f^!T_q^{r*}N$ means the

pull-back of $T^{r*}_q N$ with respect to f. Then we define $T^r_q f: T^r_q M \longrightarrow T^r_q N$ to be the dual map of $T^{r*}_q f$ and we obtain a bundle functor T^r_q with values in the subcategory $\underline{VM} \subset \underline{FM}$ of smooth vector bundles.

Let $A = E(k)/Ⓐ$ be a Weil algebra, $r,q \geq 1$ two integers. We have the following lemma.

Lemma 7.1 The following equality is satisfied:

$$\mathrm{Adm}(A, T^r_q) = \{\omega \in (J^r_0(\mathbb{R}^k, \mathbb{R}^q)_0)^* : \forall \gamma \in Ⓐ^q \ \ \omega(j^r_0 \gamma) = 0\},$$

where $Ⓐ^q = Ⓐ \times \ldots \times Ⓐ$, q-times.

Hence we have the following corollary.

Corollary 7.1 There is a bijection between the natural transformations $I: T^A \longrightarrow T^r_q$ and the set $\{\omega \in (J^r_0(\mathbb{R}^k, \mathbb{R}^q)_0)^* : \forall \gamma \in Ⓐ^q \ \omega(j^r_0 \gamma) = 0\}$. This bijection is given by $I \longrightarrow I(\mathbb{R}^k)(j^A(id_k))$.

Proof of Lemma 7.1. (a) " $\subset$ " Consider $\omega \in \mathrm{Adm}(A, T^r_q)$. Let $\gamma \in Ⓐ^q$. By Theorem 3.1 (since $j^A \gamma = j^A 0$) we have that $T^r_q \gamma(\omega) = T^r_q 0(\omega)$, i.e $\omega(j^r_0 \gamma) = T^r_q \gamma(\omega)(j^r_0(id_q)) = T^r_q 0(\omega)(j^r_0(id_q)) = \omega(j^r_0(0)) = 0$.

(b) " $\supset$ " Consider $\omega \in (J^r_0(\mathbb{R}^k, \mathbb{R}^q)_0)^*$. Suppose that $\omega(j^r_0 \gamma) = 0$ for any $\gamma \in Ⓐ^q$. Let $\varphi: \mathbb{R}^k \longrightarrow \mathbb{R}^{k+1}$ be a mapping such that $j^A \varphi = j^A i_k$. Of course $\rho \circ \varphi - \rho \circ i_k \in Ⓐ^q$ for any germ $\rho: (\mathbb{R}^{k+1}, 0) \longrightarrow (\mathbb{R}^q, 0)$. Hence $0 = \omega(j^r_0(\rho \circ \varphi - \rho \circ i_k)) = \omega(j^r_0(\rho \circ \varphi)) - \omega(j^r_0(\rho \circ i_k)) = T^r_q \varphi(\omega)(j^r_0 \rho) - T^r_q i_k(\omega)(j^r_0 \rho)$. Therefore $T^r_q \varphi(\omega) = T^r_q i_k(\omega)$, i.e $\omega \in \mathrm{Adm}(A, T^r_q)$. ∎

8. Vector spaces of natural transformations of Weil functors into linear bundle functors.

We shall start with the following definition.

Definition 8.1 A bundle functor $F: \underline{Mf} \longrightarrow \underline{FM}$ is called a linear bundle functor if $\mathrm{im}(F) \subset \underline{VM}$, where $\underline{VM}$ is the category of linear fibre bundles and their morphisms.

It is easily seen that if F is a bundle functor and G is a linear bundle functor, then the set $\mathrm{Trans}(F,G)$ of all natural transformations of F into G admits the following vector space structure: (a) $\forall\ I,J \in \mathrm{Trans}(F,G)$ $I+J \in \mathrm{Trans}(F,G)$, where $(I+J)(M): FM \longrightarrow GM$ is given by

$(I+J)(M)(v):= I(M)(v) + J(M)(v)$, and (b) $\forall \lambda\in\mathbb{R}$, $I\in\mathrm{Trans}(F,G)$ $\lambda I\in\mathrm{Trans}(F,G)$, where $(\lambda I)(M):FM\longrightarrow GM$ is defined by $(\lambda I)(M)(v):= \lambda(I(M)(v))$.

Let F be a linear bundle functor and $A = E(k)/Ⓐ$ a Weil algebra. It is easy to verify that the map J described in Theorem 5.1 is a linear isomorphism between vector spaces $\mathrm{Trans}(T^A,F)$ and $\mathrm{Adm}(A,F)$. Moreover, $\mathrm{Adm}(A,F)$ is a vector subspace of $F_0\mathbb{R}^k$. Hence we have the following corollary.

Corollary 8.1 Let F be a linear bundle functor and $A= E(k)/Ⓐ$ a Weil algebra. Then $\mathrm{Trans}(T^A,F)$ and $\mathrm{Adm}(A,F)$ are finite dimensional vector spaces and $\dim(\mathrm{Trans}(T^A,F)) = \dim(\mathrm{Adm}(A,F))\leqslant\dim(F_0\mathbb{R}^k)$.

The following example shows that there exists a linear bundle functor G such that $\dim(\mathrm{Trans}(G,G)) = \infty$.

Example 8.1 Let

$$G = \bigoplus_{q\in\mathbb{N}} \Lambda^q T$$

where T is the tangent functor, Λ^q is the inner product and $\oplus$ is the Whitney product. We see that if $q > \dim M$, then $\Lambda^q TM\simeq M\times\{0\}$ and therefore GM is finite dimensional. Consequently, G is a linear bundle functor on $\mathcal{Mf}$. For each natural number k define $I^k\in\mathrm{Trans}(G,G)$ to be the family of maps $I^k(M):GM\longrightarrow GM$ given by $I^k(M)(\{v^q\}) =\{\delta^q_k v^q\}$, where δ^q_k is the Kronecker delta. Of course, the set $\{I^k : k\in\mathbb{N}\}$ is linearly independent. Hence $\dim(\mathrm{Trans}(G,G))=\infty$.

A simple application of Corollary 8.1. We fix a natural number q . As a simple application of Corollary 8.1 we will determine all natural transformations of TT into $\Lambda^q T$. Since the classical tangent functor is the Weil functor of the algebra of dual numbers $D= E(1)/\underline{m}(1)^2$, the iterated tangent functor TT is the Weil functor of the tensor product $D\otimes D = E(2)/\langle x^2,y^2\rangle$, where x^2,y^2 is the ideal in $E(2)$ generated by germs : $x^2:\mathbb{R}^2\longrightarrow\mathbb{R}$, $y^2:\mathbb{R}^2\longrightarrow\mathbb{R}$ given by $x^2(x,y)=x^2$ and $y^2(x,y)=y^2$ (see [9] or [5]). We have two natural projections of TT onto T . Namely, $T(p^T_M):TTM\longrightarrow TM$ and $p^T_{TM}:TTM\longrightarrow TM$, $M\in\mathcal{Mf}$, where $p^T_M:TM\longrightarrow M$ is the bundle projection. It is easily seen that the above projections are natural transformations of TT into T. Let e_1, e_2 be the canonical basis of $\mathbb{R}^2$ and $T_0\mathbb{R}^2 \simeq \mathbb{R}^2$. For each $z \in \mathbb{R}^2$, we have translation by z denoted by $\tau_z : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by $\tau_z(y) = z + y$. Consider vector $v_0 = [t \longrightarrow T(\tau_{te_1})(e_2)] \in TT\mathbb{R}^2$. We see that

$$T(p^T_{\mathbb{R}^2})(v_0) = e_1 \quad \text{and} \quad p^T_{T\mathbb{R}^2}(v_0) = e_2 \ .$$

Therefore the above natural transformations of TT onto T are linearly independent. On the other hand, by Corollary 8.1, $\dim(\mathrm{Trans}(TT,T)) \leqslant \dim(T_0\mathbb{R}^2) = 2$. Hence the above natural transformations form a basis of the vector space of all natural transformations of TT into T. Now, by using Corollary 8.1 it is easy to verify that: (a) Any natural transformation of TT into $\Lambda^q T$ is the zero transformation, provided $q \geqslant 3$, and (b) Any natural transformation of TT into $\Lambda^2 T$ is of the form $\lambda\, T(p^T_M) \wedge p^T_{TM}$, $M \in \underline{Mf}$, where $\lambda \in \mathbb{R}$.

9. Proof of Proposition 1.1. ([8]) Let $F : \underline{Mf} \longrightarrow \underline{FM}$ be a bundle functor. By results of Epstein-Thurston [3], for any $n \in \mathbb{N}$ $F_n = F|\underline{Mf_n}$ is a natural bundle in dimension n. In particular, the map

$$F\tau : \mathbb{R}^n \times F\mathbb{R}^n \longrightarrow F\mathbb{R}^n, \quad (x,v) \longrightarrow F\tau_x(v)$$

(τ_x is translation by x) is a smooth action of $(\mathbb{R}^n, +)$ on $F\mathbb{R}^n$ for any natural number n. Using this fact we prove Proposition 1.1 in the following way: Let $f : M \times P \longrightarrow N$ be a smoothly parametrized family. By applying charts we can assume that $M = \mathbb{R}^m$, $N = \mathbb{R}^n$ and $P = \mathbb{R}^k$. Consider the family $\widetilde{Ff} : F\mathbb{R}^m \times \mathbb{R}^k \longrightarrow F\mathbb{R}^n$ given by $(\widetilde{Ff})_p = F(f_p)$, $p \in \mathbb{R}^k$. It is obvious that $\widetilde{Ff}$ is smoothly parametrized, provided $k=0$. So, assume that $k > 0$. One can see that $f_p = f \circ \tau_{(0,p)} \circ i$, where $i : \mathbb{R}^m \longrightarrow \mathbb{R}^m \times \mathbb{R}^k$ is given by $i(y) = (y,0)$ and $\tau_{(0,p)}$ is the translation by $(0,p) \in \mathbb{R}^m \times \mathbb{R}^k$. Hence the family $(\widetilde{Ff})_p = Ff \circ F\tau_{(0,p)} \circ Fi$, $p \in \mathbb{R}^k$ is smoothly parametrized. This ends the proof of Proposition 1.1. ∎

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