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Clifford Approach to Metric Manifolds

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1. Introduction

In recent papers we have put forward 'spin gauge theories' of the electroweak interactions and of gravitation [1,2] and of gravitation combined with the electroweak interactions [3]; these models are based upon Clifford algebras. In a further paper [4], we have outlined our progress towards including the strong interactions to produce a unified model of the four fundamental forces. These models use curved space-time to describe gravitation, and we have introduced curved manifolds in such a way as to build in the 'Clifford structure' which we require. We do not start with a general topological manifold, and then make it appropriate to our needs by building structure into it. Rather, we define a type of manifold which automatically includes the structure we need, namely, a 'Clifford manifold' with built-in metric, tensor structure *and* spin structure. While our definition is avowedly less general than standard definitions of topological manifolds, we dispense with the awkwardness of introducing a spin bundle 'associated with' a tensor manifold. Since both tensor and spin structures are used to describe nature, it is more satisfactory in physical models to use a single manifold which embodies both. Thus a Clifford manifold might provide a comprehensive model for the physical world.

The concepts underlying the Clifford manifold are not new. We freely make use ideas from Cartan [5] and from standard versions of differential geometry [6], and our ideas overlap to some extent with those of, for example, Penrose and Rindler [7], Carmeli [8], Hestenes and Sobczyk [9], Benn and Tucker [10]. However, our particular mix of ideas differs from those of other approaches in some respects, notably in the physical interpretation of the 'frame field'.

We have not regarded our view as being a particularly significant mathematical development, but on several occasions when we have lectured on spin gauge theories, and in private discussions, we have been urged to set down in writing our ideas on manifolds. This talk is an attempt at giving a coherent account of these ideas. We do not, however, see this account as anything but a sketch of our approach; we shall talk of 'assumptions' rather than 'axioms', and in general we shall not attempt to give a full and rigorous account.

The general interest in this approach stems from the fact that the majority of theories in mathematical physics are based on metric spaces and on associative algebras; these are the two features automatically included in Clifford manifolds. Thus the simplification which we achieve is because we choose to assume metricity and associativity from the beginning. Of course, there are purely topological physical theories, and there are models using, for example, octonians for which Clifford manifolds may not be appropriate.

2. Definition of Clifford Manifolds

There appear to be two fundamental features to be included in mathematical models describing the natural world. The first is the curvature of physical space arising in gravitational theories and the second is the spin of elementary particles. The Clifford manifold that we are developing automatically includes mathematical structures which represent both of these important features. The curved space is represented by a manifold and spin is introduced by imposing a Clifford structure upon it.

It is well known that an n-dimensional real manifold cannot generally be described in terms of a single non-degenerate coordinate system. We therefore adopt the familiar concept of a manifold as a union of a countable number of open sets, called 'patches', on each of which a non-degenerate coordinate system can be defined. We shall be defining what we mean by 'non-degenerate', but shall not discuss standard differential geometric concepts such as 'patch', 'chart', or 'atlas'. Our work so far is confined to local properties of manifolds; although we believe that our approach may provide a convenient method of dealing with homological and homotopic properties, we have so far made no attempt studu these applications.

We take as our starting point the set of coordinates $x \equiv \{x^\mu;\ \mu=1,2,\ldots,n\}$ which may be defined at every point x of a single patch P. Increments of the coordinates $\{x^\mu\}$ are denoted by $\{\delta x^\mu\}$ and derivatives with respect to $\{x^\mu\}$ by $\{\partial_\mu\}$. We view the increments as magnitudes only, and the derivatives purely as differential operators, that is, not as 'vectors' in any sense, as in some presentations of differential geometry. Consequently $\{\delta x^\mu\}$ and $\{\partial_\mu\}$ are both sets of commuting quantities, satisfying

$$[\delta x^\mu,\delta x^\upsilon] = 0 \tag{2.1}$$

and

$$[\partial_\mu,\partial_\upsilon] = 0. \tag{2.2}$$

On a single patch P, the coordinates x may be replaced by a new set
$y \equiv \{y^\upsilon\}$, provided that

$$y^\upsilon = Y^\upsilon(x) \qquad (\upsilon=1,2,\ldots,n) \tag{2.3}$$

are several times differentiable, with non-zero Jacobian:

$$\partial_\mu(y^\upsilon) \neq 0. \tag{2.4}$$

The increments transform by

$$dy^\upsilon = \partial_\mu(Y^\upsilon)\, dx^\mu \equiv A^\upsilon{}_\mu(x)\, dx^\mu \tag{2.5}$$

under changes of coordinate system.

Our first assumption introduces the Clifford structure which is used to define the spin structure on the manifold:

Assumption 1

A spin space is generated by the elements of a Clifford algebra the vector basis $\{\mathbf{e}_i;\ i=1,2,\ldots,n\}$ of which spans an n-dimensional linear space. An 'orthonormal' basis $\{\mathbf{e}_i\}$ satisfies

$$\{\mathbf{e}_i,\mathbf{e}_j\} \equiv \mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i = 2I\eta_{ij}, \tag{2.6}$$

where I is the unit scalar of the algebra and (η_{ij}) is the non-singular 'Minkowski' metric

$$(\eta_{ij}) = \mathrm{diag}\ (+1,..p\ \text{times}..,+1,-1,..q\ \text{times}..,-1), \tag{2.7}$$

so that the signature is (p+,q−) with p+q=n.

The spin structure is linked with each point x of a patch P of the manifold by taking the set $\{\mathbf{e}_i\}$ as a local basis for the tangent space T_x at x:

Assumption 2

The global structure of the manifold is determined by defining a vector basis $\{\mathbf{e}_\mu(x);\ \mu=1,2,\dots,n\}$ as an x-dependent linear combination of the set $\{\mathbf{e}_i\}$:

$$\mathbf{e}_\mu(x) = h_\mu{}^i(x)\ \mathbf{e}_i, \tag{2.8}$$

The coefficients $\{h_\mu{}^i(x)\}$, defining the 'vielbein field', are several times differentiable on a patch. Further, the matrix $(h_\mu{}^i(x))$ must be real and nonsingular.

The assumed differentiability of the $\{h_\mu{}^i(x)\}$ implies that on a patch the basis $\{\mathbf{e}_\mu(x)\}$ is continuous in x. Then continuity of $\{\partial_\mu y\}$ ensures that transformed bases are also continuous.

It follows that the x-dependent basis $\{\mathbf{e}_\mu(x)\}$ forms a vector basis of a Clifford algebra at each point x, since from (2.6) and (2.8)

$$\begin{aligned}\{\mathbf{e}_\mu(x),\mathbf{e}_\upsilon(x)\} &\equiv \mathbf{e}_\mu(x)\mathbf{e}_\upsilon(x)+\mathbf{e}_\upsilon(x)\mathbf{e}_\mu(x)\} \\ &= 2I\ g_{\mu\upsilon}(x),\end{aligned} \tag{2.9}$$

where

$$g_{\mu\upsilon}(x) = h_\mu{}^i(x)\ h_\upsilon{}^j(x)\ \eta_{ij}. \tag{2.10}$$

The functions $\{g_{\mu\upsilon}(x)\}$ are differentiable, and define the metric on the manifold; (2.10) is the standard relation between the metric and the vielbein field.

We have already indicated that we intend to view the increments $\{\delta x^\mu\}$ as magnitudes. By introducing the tangent space T_x, we are able to define them more precisely as magnitudes of displacements. Since the $\{\mathbf{e}_\mu\}$ are vectors in T_x, a displacement in T_x may be defined by

$$\delta \mathbf{s} = \delta x^\mu \mathbf{e}_\mu. \tag{2.11}$$

When the increments $\{\delta x^\mu\}$ become the infinitesimals $\{dx^\mu\}$, we may link this displacement in T_x to the incremental displacement on the manifold at x:

Assumption 3

There exists a choice of the set of the vielbein field $\{h_\mu{}^i(x)\}$ such that

$$d\mathbf{s} \equiv dx^\mu\ \mathbf{e}_\mu(x); \tag{2.12}$$

is the general infinitesimal displacement on the manifold. We call $d\mathbf{s}$ the 'linear metric'. At each point, the $\{dx^\mu\}$ transform in the same way as $\{x^\mu\}$ and $\{\mathbf{e}_\mu(x)\}$ transform covariantly with the increments $\{dx^\mu\}$.

This ensures that d**s** is invariant under changes of the coordinate system.

Assumptions 1 and 2 relate the algebraic structures at every point of a patch P. The spin space algebra is common to every point; in particular, the unit I is a constant element related to all points of P. We do not regard each point as possessing its own Clifford algebra; rather, there is a single algebra defined by (2.9) which varies continuously through the patch. This view of a single 'distorting algebra' accords with physical intuition: elements of a Clifford algebra, for example spinors, are taken to represent physical particles; these are extensive 'objects' in physical space-time, and are not associated with a single point x. The 'distorting algebra' view is also mathematically essential: later, we shall differentiate (2.9) and other equations containing the basis vectors $\{\mathbf{e}_\mu(x)\}$. This means that we take the limit of differences of $\mathbf{e}_\mu(x)$ at different points x; we must therefore view the vectors associated with different points as belonging to 'the same algebra' which distorts as x changes. This view is reinforced by the fact that we shall assume that $\partial_\mu I$ is zero, so that the unit I is 'the same object' at all points of the manifold.

Since the matrix $(h_\mu{}^i(x))$ is real, continuous and non-singular, the definition (2.10) ensures that the matrix (η_{ij}) is congruent to $(g_{\mu\upsilon}(x))$ at every point of P. Thus, by Sylvester's Theorem [11], the matrix $(g_{\mu\upsilon}(x))$ is non-singular, with signature (p+,q−) at all points. If the manifold represents 'normal' space-time, the assumption that the metric is non-singular is valid. However, if we wished to study black holes, it would be necessary to allow degeneracy of the metric at the boundary.

As an example we consider four-dimensional space-time, the algebra of which is usually taken to be the Dirac algebra $C_{1,3}$, with n=4, p=1 and q=3. The basis vectors $\{\mathbf{e}_\mu(x)\}$ are then represented by 4x4 matrices

$$\gamma_\mu(x) = h_\mu{}^i(x)\ \gamma_i, \qquad (2.13)$$

where $\{\gamma_i;\ i=1,2,3,4\}$ is a set of Dirac matrices. We call the set $\{\gamma_\mu(x)\}$ the 'frame field', since it represents a frame of reference at each point of the space-time manifold. In our 'spin gauge theories' of particle interactions we assume that the frame field is a *physical* field; the mass of a fermion is interpreted as an interaction between the fermion and the frame field. The basic relation (2.9) tells us that the frame field is the 'Dirac square root' of the space-time metric tensor $g_{\mu\upsilon}(x)$, which we know to be associated with the gravitational field; we have shown [2,3] that the frame field interaction does indeed give the usual Einstein-Hilbert gravitational Lagrangian, modified by a term providing a very short-range force.

The 'linear metric' of $C_{1,3}$

$$ds = \gamma_\mu(x)\ dx^\mu \tag{2.14}$$

is the incremental displacement on the space-time manifold, and

$$ds\ ds = g_{\mu\upsilon}\ dx^\mu dx^\upsilon\ I \equiv ds^2\ I, \tag{2.15}$$

defining the usual quadratic metric ds^2. The linear metric (2.14) contains more information than the quadratic metric, since it defines both the tensor metric $g_{\mu\upsilon}$ and the spin frame γ_μ; this is why the Clifford manifold is both a tensor and a spin manifold.

Since the $\{e_\mu(x)\}$ at each point x are built up from the same spin space algebra, we may deduce that the derivatives

$$\partial_\mu e_\upsilon(x) = \lim \{[e_\upsilon(x^1,..,x^\mu+\delta x^\mu,..,x^n) - e_\upsilon(x^1,..,x^\mu,..,x^n)]/\delta x^\mu\}$$

exist and are continuous. The definition (2.8) leads us to conclude that the derivatives are vectors in the spin space algebra at the point x.

3. The Connection Coefficients

A Riemannian manifold has only a tensor structure and no spin structure. Thus the covariant derivatives of quantities on a Riemannian manifold involves only a 'tensor connection', that is, they only take account of the tensor properties of the quantity being differentiated. On the other hand a Clifford manifold has both tensor and spin structures. We thus have several different types of quantity that must be differentiated: those with tensor properties only, those with spin properties only and those with both. The covariant derivatives of these quantities must take into account these various properties, and include tensor and spin connections as appropriate.

For a pure tensor quantity, the covariant derivative is that of the Riemannian manifold; for example, the covariant derivative of a rank 1 tensor field $a_\mu(x)$ is

$$D_\mu a_\upsilon = \partial_\mu a_\upsilon - \Gamma_{\mu\upsilon}{}^\rho\ a_\rho. \tag{3.1}$$

where $\Gamma_{\mu\upsilon}{}^{\rho}$ are are the 'tensor connection coefficients'. The condition $D_\mu a_\upsilon=0$ defines 'parallel transport' of rank 1 tensors on the manifold.

As an illustration of the form of the covariant derivative when both tensor and spin properties are involved, we consider the Clifford vector $\mathbf{e}_\upsilon(x)$; its covariant derivative is given by

$$D_\mu \mathbf{e}_\upsilon = \partial_\mu \mathbf{e}_\upsilon - \Gamma_{\mu\upsilon}{}^{\rho}\mathbf{e}_\rho + [G_\mu,\mathbf{e}_\upsilon], \tag{3.2}$$

where

$$[G_\mu,\mathbf{e}_\upsilon] \equiv G_\mu \mathbf{e}_\upsilon - \mathbf{e}_\upsilon G_\mu, \tag{3.3}$$

and G_μ is known as the 'spin connection'.

The inclusion in $D_\mu \mathbf{e}_\upsilon$ of the terms involving both $\Gamma_{\mu\upsilon}{}^{\rho}$ and G_μ is a result of our statement that a Clifford manifold embodies both tensor and spin structures. The tensor structure is separated out if we differentiate the basic relation (2.9) with respect to x^σ, remembering that I is constant on the manifold; then

$$\{D_\sigma \mathbf{e}_\mu(x),\mathbf{e}_\upsilon(x)\} + \{\mathbf{e}_\mu(x),D_\sigma \mathbf{e}_\upsilon(x)\} = 2\mathrm{I}\, D_\sigma g_{\mu\upsilon}(x).$$

Substituting from (3.2) and using (2.9) itself, we establish the usual

$$D_\rho g_{\mu\upsilon} = \partial_\rho g_{\mu\upsilon} - \Gamma_{\rho\mu}{}^{\sigma} g_{\upsilon\sigma} - \Gamma_{\rho\upsilon}{}^{\sigma} g_{\mu\sigma}, \tag{3.4}$$

which does not involve the spin connection.

Assumption 4

The Clifford manifold is defined by the condition that

$$D_\mu \mathbf{e}_\upsilon = 0 \quad (\upsilon=1,2,,...,n). \tag{3.5}$$

This is the condition that the vector basis is 'parallel transported' on the Clifford manifold.

As we commented at the end of the previous section the derivatives $\partial_\mu \mathbf{e}_\upsilon$ are Clifford vectors. Thus the condition (3.5) implies that

$$G_\mu = \tfrac{1}{2}\Sigma\, \mathbf{e}_i \mathbf{e}_j \eta^{ik} h_k{}^{\sigma}(D_\mu h_\sigma{}^{j}) + S_\mu \mathrm{I}, \tag{3.6}$$

where

$$D_\mu h_\sigma{}^{i} = \partial_\mu h_\sigma{}^{i} - \Gamma_{\mu\sigma}{}^{\rho}\, h_\rho{}^{i} \tag{3.7}$$

and S_μ is an arbitrary field.

By using the condition (3.5) together with the equation (2.9), we may show that the metric tensor $g_{\mu\upsilon}(x)$ satisfies

$$D_\rho g_{\mu\upsilon} \equiv 0 \qquad (3.8)$$

identically; that is, it is also parallel transported. Thus, by using (3.4) we may deduce that

$$g_{\mu\upsilon,\sigma} = \Gamma_{\mu\sigma,\upsilon} + \Gamma_{\upsilon\sigma,\mu} \qquad (3.9)$$

where

$$\Gamma_{\mu\upsilon}{}^{\rho} = g^{\rho\sigma}\,\Gamma_{\mu\upsilon,\sigma}. \qquad (3.10)$$

If the tensor connection $\Gamma_{\mu\upsilon,\sigma}$ is assumed to be symmetric in the suffixes μ,υ, (3.9) implies that it takes the usual Riemannian form.

The condition (3.5) has greater information content than the usual Riemannian condition (3.8) since we can deduce (3.8) from (3.5). This shows the advantage of employing a Clifford manifold since both the tensor and the spin connections are automatically introduced. In the same way, the linear metric d**s**, defined by (2.14), defines both the tensor and the spin structures of the manifold. The usual quadratic metric ds^2 defines only the tensor structure.It would be possible to develop this approach further, introducing dual spaces and the curvature tensor. But it is perhaps more useful to give a brief description of the group properties on M and on T_x.

4. Group properties

On a patch of a curved manifold, the allowable sets of coordinate transformations

$$y^\upsilon = Y^\upsilon(x) \qquad (4.1)$$

are those with continuous partial derivatives and non-zero Jacobian. Covariant derivatives of vector fields on the manifold are defined by (3.3). We can introduce tensor fields and their covariant derivatives in the usual way. We have seen that the Clifford vector basis $\{\mathbf{e}_\mu(x)\}$ has both vector and spin properties; it can therefore be used to define both tensor and spin transformation properties.

Sets of quantities bearing Latin suffixes have transformation properties in the flat tangent space T_x, each suffix transforming under the (generalised) Lorentz group of (pseudo-)rotations. For example, consider the basis vectors $\{\gamma_i\}$ of the Dirac algebra, satisfying

$$-\gamma_1^2 = -\gamma_2^2 = -\gamma_3^2 = \gamma_4^2 = I,$$

where I is the 4x4 unit matrix. Rotations in the (1,2) plane of T_x are generated by the bivector $\gamma_{12} \equiv \gamma_1\gamma_2$, with $\gamma_{12}^2 = -I$. The fundamental transformation is the *spin transformation*

$$\gamma_i \to \exp(-\tfrac{1}{2}\theta\gamma_{12})\ \gamma_i\ \exp(\tfrac{1}{2}\theta\gamma_{12}) \tag{4.2}$$

for i=1,2,3,4. Since this transformation is a similarity transformation, any element C of the algebra transforms in the same way:

$$C \to \exp(-\tfrac{1}{2}\theta\gamma_{12})\ C\ \exp(\tfrac{1}{2}\theta\gamma_{12}). \tag{4.3}$$

Since γ_{12} anti-commutes with γ_1 and γ_2, but commutes with γ_3 and γ_4, we can deduce the *vector transformation* on $\{\gamma_i\}$:

$$\begin{aligned} &\gamma_1 \to \gamma_1 \exp(\theta\gamma_{12}) = \gamma_1 \cos\theta - \gamma_2 \sin\theta, \\ &\gamma_2 \to \gamma_2 \exp(\theta\gamma_{12}) = \gamma_1 \sin\theta + \gamma_2 \cos\theta, \\ &\gamma_3 \to \gamma_3,\ \gamma_4 \to \gamma_4. \end{aligned} \tag{4.4}$$

Note, however, that this form of transformation is specific to vectors in T_x, whereas (4.3) is quite general.

In a very similar way, the transformation

$$C \to \exp(-\tfrac{1}{2}\theta\gamma_{14})\ C\ \exp(\tfrac{1}{2}\theta\gamma_{14}), \tag{4.5}$$

in which $\gamma_{14}^2 = I$, defines the effect of a hyperbolic rotation or 'boost' in the $\mathbf{e}_1$ direction, with hyperbolic functions replacing the circular functions.

One advantage of the Clifford approach is the immediacy of the relationship between the spin transformations, exemplified by (4.2), and the corresponding vector transformations such as (4.4). We emphasise again that the spin transformations such as (4.3) define almost immediately *all* orthogonal tensor transformations on T_x, not just vector transformations; spin transformations are thus more fundamental than tensor transformations. The relationship between (4.2) and (4.4) exemplifies the very simple way in which the spin groups are related to the orthogonal groups through the Clifford approach. Symplectic groups also arise very naturally through Clifford algebras, in particular through complex Clifford algebras.

The spin transformations ascribed to spinors are not equivalence transformations of the form (4.2)-(4.5). Spinors are defined as elements of left ideals of a Clifford algebra, and are of the form

$$\boldsymbol{\psi} = \mathbf{CP}, \tag{4.6}$$

where **P** is a fixed idempotent of the algebra. In order to preserve the idempotent structure (4.6) under transformation, the spinor transformation corresponding to (4.3) is taken to be

$$\psi \rightarrow \exp(-\tfrac{1}{2}\theta\gamma_{12})\,\psi. \tag{4.7}$$

A dual (or conjugate) spinor, denoted by φ^*, is an element of a right ideal of the algebra; to preserve the ideal structure, the transformation corresponding to (4.3) is taken to be

$$\varphi^* \rightarrow \varphi^* \exp(\tfrac{1}{2}\theta\gamma_{12}). \tag{4.8}$$

An element of the 'density matrix' $\psi\psi^*$ then transforms by the same rule (4.3) as the general element of the algebra. Since this is the only combination of spinors that occurs in quantum mechanical calculations, the transformation rules (4.7) and (4.8) do not violate the general rule (4.3) in practice.

The vielbein field $\{(h_\mu{}^i(x)\}$ is the basic link between the manifold and the spin space; as a consequence, it has transformation properties appropriate to both. The lower suffix 'μ' transforms covariantly to the upper suffix on the basis vectors γ^μ. The upper suffix 'i' transforms under the Lorentz group in the spin space [12].

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