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1. Introduction.

Since Shanks and Pursell proved in [8] that maximal ideals of the Lie algebra $\mathfrak{X}(M)$ of all smooth vector fields on a compact manifold M consist of those vector fields which are flat at a given point of M, many authors have studied ideals of different type Lie algebras of vector fields. For instance, maximal ideals of the Lie algebra $\mathfrak{X}(M)$ for non-compact M are described by Vanžura [10] in terms of the Stone-Cech compactification of M.

The algebraic approach developed in [1,3] gives the description for a larger class of the Lie algebras of vector fields, e.g. the vector fields tangent to a given foliation and for analytic cases as well. All the maximal ideals turned out to be simultaneously modules over the rings of the corresponding class of functions.

It was conjectured by Vanžura in early eighties that it is in fact true for any ideal of the Lie algebra $\mathfrak{X}(M)$.
This note contains a result describing all ideals of a larger class of the Lie algebras of vector fields (including analytic cases) which easily implies the Vanžura's conjecture.

2. Formulation of the main result.

In speaking of an n-dimensional manifold M of class $\mathcal{C}$ over the field $\mathbb{F}$, we shall mean one of three things:

This paper is in final form and no version of it will be submitted for publication elsewhere.

(a) a real paracompact smooth manifold of real dimension n, when $\mathcal{E}$ denotes C^{∞} and $\mathbb{F}$ denotes the real field $\mathbb{R}$;

(b) a real paracompact real-analytic manifold of real dimension n, when $\mathcal{E}=C^{\omega}$ and $\mathbb{F}=\mathbb{R}$;

(c) a complex manifold of complex dimension n for which each connected component is Stein, when $\mathcal{E}$ denotes the holomorphic differentiability class $\mathcal{H}$ and $\mathbb{F}$ denotes the complex field $\mathbb{C}$.

For each of (a), (b), (c), one has an embedding theorem (due to Whitney [11] in case (a), to Narasimhan [6] in case (c), and to Grauert [4] in case (b)): a connected manifold of class $\mathcal{E}$ and dimension n is $\mathcal{E}$-diffeomorphic to a closed $\mathcal{E}$-submanifold of $\mathbb{F}^{2n+1}$.

On a manifold of class $\mathcal{E}$ consider now a class $\mathcal{E}$ foliation $\mathfrak{F}$ and set $\mathfrak{X}(\mathfrak{F})$ to be the Lie algebra of class $\mathcal{E}$ vector fields on M which are tangent to the leaves of $\mathfrak{F}$. Observe that $\mathfrak{X}(\mathfrak{F})=\mathfrak{X}(M)$ for $\mathfrak{F}=\{M\}$ and that the Lie algebra $\mathfrak{X}(\mathfrak{F})$ is clearly a module over the algebra $\mathcal{E}(M)$ of the $\mathbb{F}$-valued functions of class $\mathcal{E}$ on M.

(2.1)**Theorem.** (cf.[3]) *The $\mathcal{E}(M)$-module $\mathfrak{X}(\mathfrak{F})$ is finitely generated by vector fields which span the tangent spaces to the leaves of $\mathfrak{F}$ at every point of M.*

(2.2)**Remark.** In section 4 we shall prove a stronger result using the above fact in the holomorphic case only.

The main result which immediately implies the Vanžura's conjecture is the following.

(2.3)**Theorem.** *The ideals of the Lie algebra $\mathfrak{X}(\mathfrak{F})$ are of the form $I\mathfrak{X}(\mathfrak{F}) := \mathrm{span}\{ fX : f\in I \text{ and } X\in\mathfrak{X}(\mathfrak{F})\}$ for I being $\mathfrak{X}(\mathfrak{F})$-invariant ideals of the associative algebra $\mathcal{E}(M)$.*
In fact, the assignment $I\longmapsto I\mathfrak{X}(\mathfrak{F})$ establishes a one-one correspondence between $\mathfrak{X}(\mathfrak{F})$-invariant ideals of $\mathcal{E}(M)$ and Lie ideals of $\mathfrak{X}(\mathfrak{F})$. The inverse mapping has the form $\mathfrak{K}\longmapsto\mathfrak{K}(\mathcal{E}(M)) :=\mathrm{span}\{X(f): X\in\mathfrak{K} \text{ and } f\in\mathcal{E}(M)\}$.

The $\mathfrak{X}(\mathfrak{F})$-invariance of I means obviously that $\mathfrak{X}(\mathfrak{F})(I)\subseteq I$, i.e. that $X(f)\in I$ for each $X\in\mathfrak{X}(\mathfrak{F})$ and $f\in I$. The above result covers all what was known in the subject. For example, we get immediatelly the description of maximal ideals of $\mathfrak{X}(M)$ for compact M, since maximal ideals in $\mathscr{C}(M)$ invariant with respect to derivations consist of fuctions which are flat at a given point. For M being additionally connected this shows in the real-analytic case that $\mathfrak{X}(M)$ is simple (cf. [1]).
The next two sections we devote to the proof of the main theorem.

3. Algebraic preparations.

Throughout this section $\mathscr{A}$ denotes an abelian associative unital algebra over a field $\mathbb{k}$ of characteristic $\neq 2$ and $\mathfrak{L}$ is a subalgebra of the Lie algebra $Der(\mathscr{A})$ (with the commutator bracket) of the derivations of $\mathscr{A}$. Note that $Der(\mathscr{A})$ is an $\mathscr{A}$-module in a natural way and that the Lie- and the $\mathscr{A}$-module structures are connected via the well known identity

(3.1) $[fX,gY]=fX(g)Y-gY(f)X+fg[X,Y]$,

where $f,g\in\mathscr{A}$, $X,Y\in Der(\mathscr{A})$.
In our geometrical model: $\mathscr{A}=\mathscr{C}(M)$ and $\mathfrak{L}=\mathfrak{X}(\mathfrak{F})$.
We shall call a subalgebra $\mathfrak{L}$ of $Der(\mathscr{A})$ *modular* iff it is simultaneously an $\mathscr{A}$-submodule and *strongly nowhere-vanishing* iff $\mathfrak{L}(\mathscr{A}):=span\{X(f): X\in\mathfrak{L},\ f\in\mathscr{A}\}$ equals $\mathscr{A}$.
Note that a modular Lie algebra $\mathfrak{L}$ of vector fields of class $\mathscr{C}$ is strongly nowhere-vanishing exactly if there is a finite set of vector fields from $\mathfrak{L}$ with no common zeros (c.f. [3]), that should explain the name.
Thus, by (2.1), the Lie algebra $\mathfrak{X}(\mathfrak{F})$ is a modular and strongly nowhere-vanishing subalgebra of $Der(\mathscr{C}(M))$ except for the case of $\mathfrak{F}$ being 0-dimensional.

(3.2)**Theorem.** *If $\mathfrak{L}$ is a modular strongly nowhere-vanishing subalgebra of $Der(\mathscr{A})$, then*
(a) $X(\mathscr{A})\mathfrak{L}\subseteq[\mathfrak{L},[\mathfrak{L},X]]$ for every $X\in Der(\mathscr{A})$;
(b) given a Lie ideal $\mathfrak{R}$ of $\mathfrak{L}$, $\mathfrak{R}$ is modular if and only if $[\mathfrak{L},\mathfrak{R}]\subseteq\mathfrak{R}$.

(3.3)**Remark.** We hope that the notation is clear. For instance, $X(\mathcal{A})\mathfrak{L} := \mathrm{span}\{ X(f)Y : f\in\mathcal{A},\ Y\in\mathfrak{L} \}$, etc.
The part (a) of the above theorem is due to Skriabin [8].

Proof of (3.2). To prove (a) we briefly sketch the proof of Skriabin [8].
Take $Y\in\mathfrak{L}$. For $f,g\in\mathcal{A}$ set $B(f,g)=[fY,[gY,X]]$. It is a matter of simple calculations using (3.1) to show the identity

$$B(fg^2,1)-2B(fg,g)+B(f,g^2)=-4fY(g)X(g)Y ,$$

so $fY(g)X(g)Y\in[\mathfrak{L},[\mathfrak{L},X]]$. After the linearization with respect to the variable Y, we have

$$fY(g)X(g)Z+fZ(g)X(g)Y\in[\mathfrak{L},[\mathfrak{L},X]]$$

for all $f,g\in\mathcal{A}$ and $Y,Z\in\mathfrak{L}$. Putting $f:=Y(g)$ we get

$$(Y(g))^2X(g)Z\in[\mathfrak{L},[\mathfrak{L},X]],$$

and further (after $g:=f+g$, $Z:=Y(g)Z$) in a similar way

$$(Y(g))^3X(f)Z\in[\mathfrak{L},[\mathfrak{L},X]] \text{ for all } f,g\in\mathcal{A} \text{ and } Y,Z\in\mathfrak{L}.$$

The last implies that the radical of the largest ideal J such that $JX(\mathcal{A})\mathfrak{L} \subseteq [\mathfrak{L},[\mathfrak{L},X]]$ includes $\mathfrak{L}(\mathcal{A})$ and, since $\mathfrak{L}(\mathcal{A})=\mathcal{A}$, (a) follows.
To prove (b), consider a Lie ideal $\mathfrak{K}$ of $\mathfrak{L}$ and set $\mathcal{K}=[\mathfrak{L},\mathfrak{K}]$. By (a), $\mathfrak{K}(\mathcal{A})\mathfrak{L}\subseteq[\mathfrak{L},[\mathfrak{L},\mathfrak{K}]]\subseteq\mathcal{K}$ and hence, due to (3.1),

$$\mathcal{A}\mathcal{K}=\mathcal{A}[\mathfrak{K},\mathfrak{L}]\subseteq[\mathfrak{K},\mathcal{A}\mathfrak{L}]+\mathfrak{K}(\mathcal{A})\mathfrak{L}\subseteq[\mathfrak{K},\mathfrak{L}]+\mathcal{K}\subseteq\mathcal{K},$$

so $\mathcal{K}$ is modular. In particular, $\mathfrak{K}$ is modular assuming $\mathfrak{K}=\mathcal{K}$.
Conversely, suppose that $\mathfrak{K}$ is modular. Then

$$\mathfrak{K}=\mathfrak{L}(\mathcal{A})\mathfrak{K}\subseteq[\mathfrak{L},\mathcal{A}\mathfrak{K}]+\mathcal{A}[\mathfrak{L},\mathfrak{K}]\subseteq[\mathfrak{L},\mathfrak{K}]+\mathcal{A}\mathcal{K}\subseteq\mathcal{K} \quad .$$

The inclusion $\mathcal{K}\subseteq\mathfrak{K}$ is trivial and (b) follows. ∎

It may however happen that even for finitely generated modular nowhere-vanishing $\mathfrak{L}$ there are ideals $\mathfrak{K}$ of $\mathfrak{L}$ which are larger than $[\mathfrak{L},\mathfrak{K}]$ and therefore, by (3.2), are not modular, so we can not prove the Vanžura's conjecture in this general algebraic setting.

(3.4)**Example.** Let $h:\mathbb{R}\longrightarrow\mathbb{R}$ be smooth, non-zero, but flat at 0. For $\mathcal{A}=C^{\infty}(\mathbb{R}^2)$ the $\mathcal{A}$-module $\mathfrak{L}$ generated in $\mathfrak{X}(\mathbb{R}^2)$ by the vector fields $X(x,y)=\partial_x$ and $Y(x,y)=h(y)\partial_y$ is a finitely generated modular and strongly nowhere-vanishing Lie algebra of vector fields on $\mathbb{R}^2$. The Lie ideal $\mathfrak{K}$ generated in $\mathfrak{L}$ by $\{Y\}$ is not modular. Indeed, let I be the ideal of $\mathcal{A}$ generated by the derivatives $\{\partial_y^i(h): i=0,1,\dots\}$ (it is non-trivial, since h is flat at 0). The ideal I is clearly $\mathfrak{L}$-invariant and, since $[\mathfrak{L},Y]\subseteq I\mathfrak{L}$, we have $[\mathfrak{L},\mathfrak{K}]\subseteq I\mathfrak{L}$. But $Y\notin I\mathfrak{L}$, so $[\mathfrak{L},\mathfrak{K}]\neq\mathfrak{K}$.

We shall say that a subalgebra $\mathfrak{L}$ of $\mathrm{Der}(\mathcal{A})$ has *the reproduction property* iff $X\in X(\mathcal{A})\mathfrak{L}$ for every $X\in\mathfrak{L}$.
In other words, $\mathfrak{L}$ has the reproduction property if and only if every element $X\in\mathfrak{L}$ can be written in the form $X=\sum X(f_i)Y_i$ for some finite sets $\{f_i\}\subseteq\mathcal{A}$, $\{Y_i\}\subseteq\mathfrak{L}$.

(3.5)**Theorem.** *If $\mathfrak{L}$ is a modular strongly nowhere-vanishing Lie subalgebra of $\mathrm{Der}(\mathcal{A})$ with the reproduction property, then every Lie ideal $\mathfrak{K}$ of $\mathfrak{L}$ is modular and has the form $I\mathfrak{L}$ for the ideal I of $\mathcal{A}$ equal to $\mathfrak{K}(\mathcal{A})$.*

<u>Proof.</u> Due to the reproduction property we have $\mathfrak{K}\subseteq\mathfrak{K}(\mathcal{A})\mathfrak{L}$ and by (3.2)(a) we get $\mathfrak{K}(\mathcal{A})\mathfrak{L}\subseteq[\mathfrak{L},\mathfrak{K}]\subseteq\mathfrak{K}$, so $[\mathfrak{L},\mathfrak{K}]=\mathfrak{K}(\mathcal{A})\mathfrak{L}=\mathfrak{K}$ and $\mathfrak{K}$ is modular by (3.2)(b). ∎

4. The reproduction property.

In view of (3.5), to prove the main theorem it suffices to prove the following.

(4.1).**Theorem.** *The Lie algebra $\mathfrak{X}(\mathfrak{F})$ has the reproduction property.*

<u>Proof.</u> It suffices to deal with connected manifolds. Let start with the holomorphic case. Put $\mathcal{S}$ to be the sheaf of germs of holomorphic vector fields on the Stein manifold M wich are tangent to the leaves of $\mathfrak{F}$. In view of (2.1) it is easily seen that $\mathcal{S}$ is a coherent analytic sheaf over M globally finitely generated by some sections $Y_1,\dots,Y_r\in\mathfrak{X}(\mathfrak{F})$. Due to the embedding theorem of Narasimhan [6], we have a holomorphic embedding $f=(f_1,\dots,f_s):M\longrightarrow\mathbb{C}^s$. Take $X\in\mathfrak{X}(\mathfrak{F})$ and consider the sheaf $\mathcal{R}=\{(\hat{g},\hat{g}_{ij})\in\mathcal{O}^{rs+1}:\ \hat{g}\hat{X}-\sum\hat{g}_{ij}(X(f_i)Y_j)\hat{}\ \}\subseteq\mathcal{O}^{rs+1}$, where $\hat{X}$,

etc., stand for germs of corresponding vector fields and $\mathcal{O}$ denotes the sheaf of holomorphic functions. $\mathcal{R}$ is coherent as the sheaf of relations between sections of a coherent sheaf. Since in a neighborhood U(p) of each point $p\in M$ some of the functions $\{f_i\}$ form local coordinates and some of vector fields $\{Y_j\}$ freely generate the analytic sheaf $\mathcal{F}_{|U(p)}$, it is not hard to see that X can be locally written in the form

$$X = \sum g_{ij} X(f_i) Y_j \tag{4.2}$$

with holomorphic $g_{ij}\in\mathcal{O}(U(p))$. This means that the sheaf homomorphism $\rho:\mathcal{R}\longrightarrow\mathcal{O}$ given by $(\hat{g},\hat{g}_{ij})\longmapsto\hat{g}$ is surjective. In the short exact sequence $Ker(\rho)\longrightarrow\mathcal{R}\longrightarrow\mathcal{O}$ the sheaves $\mathcal{O}$, $\mathcal{R}$, and hence $Ker(\rho)$ are coherent, and since in the long exact sequence $\Gamma(Ker(\rho))\longrightarrow\Gamma(\mathcal{R})\longrightarrow\Gamma(\mathcal{O})\longrightarrow H^1(Ker(\rho))\longrightarrow\dots$ the cohomology group $H^1(Ker(\rho))$ is trivial by the Theorem B of Cartan, ρ induces a surjective homomorphism on the level of sections, i.e. X can be written in the form (4.2) globally.

The same argument may be applied in the real-analytic case, where Tognoli [9] has pointed out the validity of Theorem B , and in smooth case, where the corresponding sheaves are soft. There is however a simpler direct geometrical proof in this cases.

Consider an embedding $f=(f_1,\dots,f_s):M\longrightarrow\mathbb{R}^s$. In $\mathbb{R}^s$ we have the canonical coordinates $x_1,\dots,x_s$, the canonical coordinate vector fields $\partial_1,\dots,\partial_s$, and the canonical scalar product. Considering M as a submanifold of $\mathbb{R}^s$, define the vector fields $Y_i\in\mathfrak{X}(\mathfrak{F})$ by $Y_i(p)=P_p(\partial_i)$, $i=1,\dots,s$, where P_p is the orthogonal projection of $\mathbb{R}^s$ onto the tangent space $T_p\mathfrak{F}_p\subseteq\mathbb{R}^s$ of the leaf $\mathfrak{F}_p$ at $p\in M$.

Take $X\in\mathfrak{X}(\mathfrak{F})$. Considering X(p) as a vector of $\mathbb{R}^s$, we have $X(p)=\sum h_i(p)\partial_i$ for some $h_i(p)\in\mathbb{R}$, $i=1,\dots,s$. Applying P_p to both sides, we get $X(p)=\sum h_i(p)Y(p)$. On the other hand,

$h_i(p)=X(p)(x_i)=X(x_i)(p)=X(f_i)(p)$, so $X=\sum X(f_i)Y_i$.

This proof does not work in the holomorphic case, since complex scalar product is not holomorphic. ∎

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