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ON THE Q-DEFORMED HEISENBERG UNCERTAINTY RELATIONS AND DISCRETE TIME

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1 Introduction

Non-commutative geometry and quantum groups are of relevance of space-time quantization and discretization.

The idea of quantization of space-time using noncommutative coordinates like

$$x_\mu x_\nu - x_\nu x_\mu = i\hbar g_{\mu\nu} \ , \quad x_\mu x_\nu - q x_\nu x_\mu = 0 \tag{1.1}$$

was presented half century ago [1].

It is natural to attempt to relate the non-commutativity parameter $q < 1$ to the minimal uncertainty in length measurement

$$\delta x > l_{Pl.} = \sqrt{\frac{2\kappa\hbar}{c^3}} \sim 10^{-35}\, m \ , \tag{1.2}$$

or time measurement

$$\delta t > \tau_{Pl.} = \frac{l_{Pl.}}{c} \sim 10^{-43}\, s \ . \tag{1.3}$$

* "This paper is in final form and no version of it will be submitted for publication elsewhere".

where κ, $\hbar$ are the gravitational and Planck constants and c is the light velocity.

Also the ideas of elementary length, nonlocal "particles" and the general theory of relativity are old [2]. There the contradiction with continual Riemannian geometry exists.

There is unclear in the continuous space-time what is the "quantum line" because a coordinate always commutes with itself.

Quite different situation is in the case of the discrete space-time or of grassmanian variables.

Here we present possible discretization on the following bases:

1) fractional supersymmetry and paragrassmanian q-deformed superspace,

2) a model, with q-deformed Heisenberg uncertainty relation[3] for the null sector.

At this moment is no known basic principle requiring space or time to be continuous or forbidding limitations on their units.

The article is organised as follows:

we start in Sect. 2 from the fractional supersymmetry and the q-deformed quantum mechanics (QM)[4] to obtain fractional superspace, which can be extended. This is done to show the possibility for obtainining the richer structure in the fractional superspace and that the base of the quantization can be done on the level of such superspace in the general case.

In Sect. 3 we present basic information about quantum cryptography (QC) (see ref. [6, 7]). as a candidate for the verficacation of the q-deformation of QM in the null sector.

In Sect. 4 we present a violation of quantum channel via q-deformation and in Sect. 5 we present the q-deformed Heisenberg uncertainty relation in QC and a model of the discretization of the space and time. The quantum space-time and the connection with the q-deformed quantum mechanics is discussed in Sect. 6.

2 Quantum superspace

We introduce supersymmetry (SUSY) with superspace $\{t, \Theta\}$, where t is the time variable and Θ a Grassmann variable i.e. $\Theta^2 = 1$.

We define the supercoordinate

$$X(t,\Theta) = x_{(0)}(t) + i\,\Theta\, x_{(1)}(t) \ , \tag{2.1}$$

where $x_{(0)}(t)$ is the ordinary commuting space coordinate (Bose or null sector variable) and $x_{(1)}(t)$ is the real anticommuting variable (Grassmann, Fermi or one-sector).

The changes of $x_{(0)}(t)$ and $x_{(1)}(t)$ follows from:

$$\delta X(t,\Theta) = X(t',\Theta') - X(t,\Theta) = i\,\varepsilon\, Q X(t,\Theta) \ , \tag{2.2}$$

where SUSY generator

$$Q = \frac{\partial}{\partial\Theta} + i\,\Theta \frac{\partial}{\partial t} = \partial_\Theta + i\,\Theta\,\partial_t \tag{2.3}$$

and ε is the infinitesimal Grassmann parameter.

We can see :

$$\begin{aligned} \partial X &= \varepsilon\,\partial_\Theta(x_{(0)} + i\,\Theta\, x_{(1)}) + i\,\varepsilon\,\Theta\,\partial_t(x_{(0)} + i\,\Theta\, x_{(1)}) \\ &= i\,\varepsilon\, x_{(1)} + i\,\varepsilon\,\Theta\,\partial_t\, x_{(0)} \end{aligned} \tag{2.4}$$

and SUSY transformations for the coordinates $x_{(0)}$ and $x_{(1)}$:

$$\delta x_{(0)} = i\,\varepsilon\, x_{(1)} \ , \quad \delta x_{(1)} = \varepsilon\,\partial_t\, x_{(0)} \ , \tag{2.5}$$

It follows immediately:

$$Q^2X = Q\,[i\,x_{(1)} + i\,\Theta\,\partial_t\,x_{(0)}] = i\,\partial_t\,(x_{(0)} + i\,\Theta\,x_{(1)}) = i\,\partial_t\,X\ , \tag{2.6}$$

$$\text{or} \qquad \frac{1}{2}\{Q,Q\}X \equiv HX = i\,\partial_t\,X\ , \tag{2.7}$$

which suggests that the Hamiltonian of the system be defined as $H = \frac{1}{2}\{Q,Q\}$ and the time translation is simply the Hamiltonian $H = i\,\partial_t$.

In this sense the $N = 1$ SUSY (it means one Grassmann Θ) is the square root of the time translation.

Let us now to turn to the general case i.e. the F-th roots of the time translation $F = 1, 2, \ldots$.

We need F real Grassmann coordinates $x_{(j)}(t)$, $j = 0, 1, \ldots, F-1$, which belong to the following $(F-j)$-sectors $x_{(j)}(t)$ and the null sector $x_{(0)}(t) \equiv x(t)$ i.e. ordinary coordinate.

These sectors can be viewed as the components of a quantum superspace with fractional SUSY [4].

We denote fractional quantum superspace

$$X^F(t,\Theta) = \sum_{j=0}^{F-1} x_{(j)}(t)\,\Theta^j = x_{(t)} + \sum_{j=1}^{F-1} x_{(j)}(t)\,\Theta^j\ , \tag{2.8}$$

where Θ is a real paragrassmann variable satisfying $\Theta^F = 0$.

Let us introduce the q-commutation relation

$$x_{(j)}(t)x_{(F-j)}(t) = q^j x_{(F-j)}(t)x_{(j)}\ . \tag{2.9}$$

In this sense the parameter q^j connects different sectors.

Then fractional SUSY has the form:

$$\delta x_{(j-1)} = i\,\varepsilon\,\alpha\,(1-q^j)\,x_{(j)}\ , \tag{2.10a}$$

$$\delta x_{(F-1)} = \varepsilon\,(F\alpha^{F-1})^{-1}\,\partial_t\,x_{(0)}\ , \tag{2.10b}$$

where α is a free constant.

We have

$$\delta^F x_{(j)}(t) = i^{(F-1)}\varepsilon_1 \dots \varepsilon_F \, \partial_F \, x_{(j)}(t) \tag{2.11}$$

since $\prod_{j=1}^{n}(1-q^j) = F$ and $\varepsilon \, x_{(j)}(t) = q^{-j} x_{(j)}(t) \, \varepsilon$.

An action invariant under (2.10) is

$$S = \int dt \, \frac{1}{2}\Big[(\partial_t \, x)^2 + i\,(F\alpha^F) \sum_{j=0}^{F-1}(1-q^{-j})\,(\partial_t \, x_{(j)})\, x_{(F-j)}\Big] \tag{2.12}$$

and fractional SUSY quantum mechanics (SSQM) of order F is defined through the algebra

$$Q^F = H \ , \quad [H,Q] = 0 \ , \quad F = 2,3,\dots \ ,$$

where H is the Hamiltonian.

The fractional SUSY can be extended by the following.way:

For the $N = 2$ SUSY, the superspace is (t, Θ_1, Θ_2) and SUSY transformation:

$$\Theta_l' = \Theta_l + \varepsilon_l \ , \ l = 1,2 \ ; \qquad t' = t + i\,\varepsilon_1 \Theta_1 + i\,\varepsilon_2 \Theta_2 \ , \tag{2.13}$$

which keep the element $dt - i\,\Theta_1 \, d\Theta_1 - i\,\Theta_2 \, d\Theta_2$ invariant.

We can combine the grassmannian variables to the complex variables $\Theta = \frac{1}{\sqrt{2}}(\Theta_1 - i\,\Theta_2)$ and $\overline{\Theta} = \frac{1}{\sqrt{2}}(\Theta_1 + i\,\Theta_2)$.

Superspace coordinate has the following form:

$$X(t,\Theta,\overline{\Theta}) = x_{(0)}(t) + i\,\overline{\Theta}\, x_{(1)}(t) + i\,\Theta\, \overline{x}_{(1)}(t) + \Theta\,\overline{\Theta}\, R_{(0)}(t) \tag{2.14}$$

and the SUSY transformations on coordinates are:

$$\begin{aligned} \delta\, x_{(0)}(t) &= i\,\varepsilon\, \overline{x}_{(1)}(t) + i\,\overline{\varepsilon}\, x_{(1)}(t) \ , \ \delta\, R_{(0)}(t) = \varepsilon\, \partial_t\, \overline{x}_{(1)}(t) - \overline{\varepsilon}\, \partial_t\, x_{(1)}(t) \ , \\ \delta\, \overline{x}_{(1)}(t) &= -\overline{\varepsilon}\,(\partial_t\, x_{(0)}(t) + i\, R_{(0)}(t)) \ , \ \delta\, x_{(1)}(t) = -\varepsilon\,(\partial_t\, x_{(0)}(t) - i\, R_{(0)}(t)) \ . \end{aligned}$$

We can introduce paragrassmann variables (for $N = 2,3,\dots$, $l = 1,\dots,N$) :

$$\Theta_l^F = 0 = \partial_l^F \ , \ \partial_l \equiv \frac{\partial}{\partial\Theta_l} \ , \ F = 1,2,\dots \ , \ (\Theta^{F-1} \neq 0 \neq \partial_l^{F-1}) \tag{2.15}$$

and the q-commutators

$$[\partial_l, \Theta_l] \equiv \partial_l \, \Theta_l - q_l \, \partial_l \, \Theta_l = \alpha_l \, (1 - q_l) \ , \tag{2.16}$$

where α_l are free parameters and $q_l \in C$ are primitive F-th roots of unity $\{q_l^F = 1$ and $q_l^n \neq 1$ for $0 < n < F\}$.

The definition (2.16) implies the derivatives:

$$\partial_l \, \Theta_l^n = \alpha_l \, (1 - q_l^n) \, \Theta_l^{n-1} + q_l^n \, \Theta_l^n \ . \tag{2.17}$$

A matrix realizations of Θ_l and ∂_l are given by

$$\Theta_l = \begin{pmatrix} 0 & a_1^l & \cdot & \cdots & 0 \\ \cdot & \cdot & a_2^l & & \cdot \\ \cdot & & & \ddots & \cdot \\ \cdot & & & & a_{F-1}^l \\ 0 & \cdot & \cdot & \cdots & 0 \end{pmatrix} , \quad \partial_l = \begin{pmatrix} 0 & \cdot & \cdot & \cdots & 0 \\ b_1^l & \cdot & & & \cdot \\ \cdot & b_2^l & & & \cdot \\ \cdot & \cdot & \ddots & & \cdot \\ 0 & \cdot & \cdot & b_{F-1}^l & 0 \end{pmatrix} \tag{2.18}$$

with the constraint $a_k^l \, b_k^l = \alpha_l \, (1 - q_l^k)$ (no summation on k).

Extended fractional SSQM , $N = 2, 3, \ldots$ is given via

$$Q_N^F \equiv H_N \ , \ [H_N, Q_N] = 0 \ , \ F = 2, 3, \ldots ,$$

where H_N are Hamiltonians and Q_N fractional supercharges. The following relations are valid as usual:

$$\left.\begin{array}{l} a_N = \frac{1}{\sqrt{2}} \left[p + i \, W_N(x)\right] \ , \ a_N^+ = \frac{1}{\sqrt{2}} \left[p - i \, W_N(x)\right] \\ [a_N^+, a_N] = \partial_x W_N(x) \equiv W_N'(x)] \\ Q_N = \partial_N^{F-1} a_N + e_N P_N \, \Theta_N a_N^+ + (1 - P_N) \, \Theta_N \\ H_N = \frac{1}{2} \, (p^2 + W_N^2) + W_N' (P_N - \frac{1}{2}) \end{array}\right\} \tag{2.19}$$

where

$$P_N = e_N \, \Theta_N^{F-1} \partial_N^{F-1} \ , \ P_N^2 = P_N$$

and the order-dependent constant e_N is given by $e_N = (F\alpha_N^{F-1})^{-1}$.

In the extended fractional SSQM appears a new interesting effect of the fractionalization of vacuum state.

The "fermion" number operator and the derivation of the formula are analogous as in [5]:

$$\Delta_N = 1 - F\,\alpha_N^F \sum_{j=0}^{F-1} (1-q^j)\, x_{N(j)}\, \partial_t\, X_{N(F-j)} \ , \tag{2.20}$$

which counts the states between parasuperpartners.

This effect can be interesting from the point of view some cosmological models, which start from the expansion of the vacuum state.

Now we will show how to look for the evidence of the q-deformed QM in the null sector in our physical world.

3 Basic information about quantum cryptography

We look for the evidence of the q-deformed QM via QC, which is based on non-deformation of QM laws.

QC [6], the candidate for key transmission in such a way that nothing could intercept it, is based on the existence of quantum properties that are incompatible in the sense that measuring one property necessarily randomizes the value of the other.

One of the QC cryptographic schemes relies on the uncertainty principle of quantum mechanics and has been demonstrated experimentally.

Here the possible experimental verification of the q-deformation of quantum mechanics is presented via measuring the validity of Heisenberg uncertainty principle using the interferometric quantum cryptographic apparatus.

One of the simplest version of this device consists of (see scheme on the Fig.1) : "single-photon" laser source, one Mach-Zehnder interferometer with optical fibre, two beam-splitters and phase modulators for the realisation minimal Bennet-Brassard cryptographic schema, which is called BB 92 in [7].

Quantum cryptographic communication with the BB 92 protocol is based on two users, say $\mathcal{A}$ and $\mathcal{B}$ who share no secret information at the outset, together with an adversary $\mathcal{E}$ who eavesdrops on their communication.

Our experimental prototype of this device, which is constructed, consists of:

1. laser diode from Seaster Optics production with driver, type AVO-A-C from Avtech Electrosystem production and attenuator;

2. single-mode optical fibres for the wavelength $830nm$, type $5/125\mu m$ of 3M production;

3. fused fibre optic splitters for wavelength $830nm$ with excess losses less then 0.5 dB from OZ Optics Ltd. production;

4. phase modulators, type APE PM-0.8-0.5-50-1-1-C from Uniphase Telecomunication Products;

5. photoncounter, type 200 MHz Photon Counting System with the 32k Data Buffer and time resolution $5ns$, from Fast Com Tec production;

6. detector for the photoncounter, type SPMCM AQ-142-FL (50 dark counts per second) from RBM GmbH production.

We have own software source generating pseudorandom binary sequences where corresponding probability is in interval $(\frac{1}{2} \pm 7.10^{-5})$.

In this configuration of quantum cryptographic device we expect the error rate less then 10^{-2}.

For the description of the QC apparatus we have the input and output of creation and anihilation operators, which describe quantum states in apparatus,

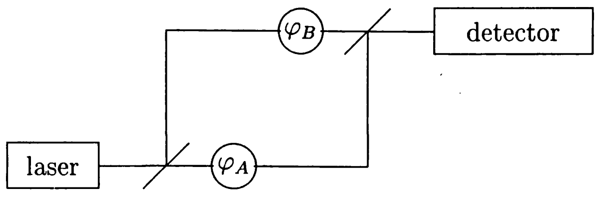


Figure 1: Schematic representation of the interferometric QC apparatus

$$a_{out}^{(1)} = 2^{-\frac{1}{2}}(a_{in}^{(1)} + ia_{in}^{(2)}) \ , \tag{3.1}$$

$$a_{out}^{(2)} = 2^{-\frac{1}{2}} \exp{(i\varphi_A)}(a_{in}^{(2)} + ia_{in}^{(1)}) \ , \tag{3.2}$$

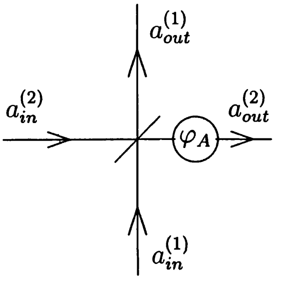
which satisfy the following algebra:

$$[a_{in}^{(1)}, a_{in}^{(2)}] = [a_{in}^{(1)}, a_{in}^{(2)+}] = [a_{in}^{(1)+}, a_{in}^{(2)}] = 0 \ , \tag{3.3}$$

$$[a_{in}^{(1)}, a_{in}^{(1)+}] = [a_{in}^{(2)}, a_{in}^{(2)+}] = 1 \ , \tag{3.4}$$

$$a_{in}^{(1)} \left|0\right\rangle = a_{in}^{(2)} \left|0\right\rangle = 0 \ , \tag{3.5}$$

with ordinary commutation relation, where $\mid 0\rangle$ means vacuum state and φ_A phase shift determined by the phase modulator of the user $\mathcal{A}$ (see Fig. 1).



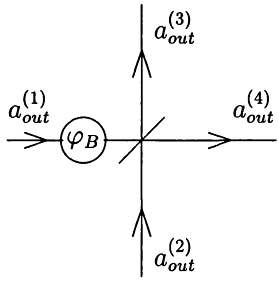


Figure 2: Schematic representation of the input and output modes at each beam-splitter of a MZ interferometer with the phase modulators determined by users $\mathcal{A}$ and $\mathcal{B}$.

In the schematic representation of QC apparatus on Fig.1 we assume the input of the beam-splitter $\mathcal{A}$ is one-photon state, produced by attenuated laser source: This state gives for the user $\mathcal{A}$ the possibility to prepare two non-orthogonal states for the cryptographic protocol BB 92 , namely:

$$\text{for } \varphi_A = 0 \qquad |\uparrow\rangle \equiv 2^{-\frac{1}{2}}(a_{out}^{(1)+} + a_{out}^{(2)+})\,|0\rangle), \tag{3.6}$$

$$\text{for } \varphi_A = \frac{\pi}{2} \qquad |\rightarrow\rangle \equiv 2^{-\frac{1}{2}}(a_{out}^{(1)+} - a_{out}^{(2)+})\,|0\rangle). \tag{3.7}$$

The user $\mathcal{B}$ has on the "in" port the "out" states comming from $\mathcal{A}$ phase shift (see Fig. 2) by φ_B determined him. In this way $\mathcal{B}$ creates the anihilation operators:

$$a_{out}^{(3)} = 2^{-\frac{1}{2}}(i\exp(i\varphi_B)\,a_{out}^{(1)} + a_{out}^{(2)}), \tag{3.8}$$

$$a_{out}^{(4)} = 2^{-\frac{1}{2}}(\exp(i\varphi_B)\,a_{out}^{(1)} + ia_{out}^{(2)}). \tag{3.9}$$

If the detector is placed on the outcoming port of the beam-splitter of the of MZ interferometer, then the detection of a photon corresponds to a projection on the following states:

$$\text{for } \varphi_B = \pi \qquad |\downarrow\rangle \equiv 2^{-\frac{1}{2}}(a_{out}^{(2)+} + ia_{out}^{(1)+})\,|0\rangle, \tag{3.10}$$

$$\text{for } \varphi_B = \frac{3\pi}{2} \qquad |\leftarrow\rangle \equiv 2^{-\frac{1}{2}}(a_{out}^{(1)+} + a_{out}^{(2)+})\,|0\rangle, \tag{3.11}$$

which are eigenstates of spin operators σ_z , resp. σ_x.

The probability of meassuring the one photon state, after eliminating all technical and other physical disturbances of the QC device on Fig. 1, by the given detector is

$$P_B = \cos^2(\frac{\varphi_A - \varphi_B}{2}) \ . \tag{3.12}$$

When $\mathcal{A}$ and $\mathcal{B}$ are using $(\varphi_A, \varphi_B) = (0, \frac{3\pi}{2})$ for the coding logical "0" and $(\varphi_A, \varphi_B) = (\frac{\pi}{2}, \pi)$ for "1", they can use BB 92 scheme, which is based on the Heisenberg uncertainty principle.

4 Violation of quantum channel via q-deformation

Let us consider two legitimate users $\mathcal{A}$ and $\mathcal{B}$ of a quantum channel without eavesdropper $\mathcal{E}$ operating on the QC device.

There are two possibilities of measuring the q-deformation on the quantum channel (i.e. optical fibre), which is a part of the QC device: 1) measuring the squeezed light and determining a deformation parameter q. In our laboratory prototype of QC apparatus we can exclude such squeezing;

2) measuring a q-deformation of the Heisenberg uncertainty relation.

We shall concern on the second case and we assume that we have no squeezed photons on the optical channel and that we eliminated all possible errors and disturbances on the QC device. It can be done because our QC device works on very short distance and is without danger noise on the quantum channel.

Such QC device, which works with BB 92 protocol, quarantees high accuracy measure that no eavesdropper is present. This accuracy is the measure validity of Heisenberg uncertainty principle and QM.

Let us suppose the violation of quantum mechanics appears as an eavesdropper measuring in the basis θ_q. If $\mathcal{A}$ sends the state $\mid\uparrow\rangle$ then the expansion of this state in the basis θ_q is

$$\mid\uparrow\rangle = \cos(\theta_q/2)\mid\uparrow\rangle_{\theta_q} - \sin(\theta_q/2)\mid\downarrow\rangle_{\theta_q}\ . \tag{4.1}$$

We shall expect $\theta_q \doteq 0$, so channel transmits the state $\mid\uparrow\rangle_{\theta_q}$ on to $\mathcal{B}$, whom we assume to measure in the σ_x basis:

$$\begin{aligned}\mid\uparrow\rangle_{\theta_q} &= \frac{1}{\sqrt{2}}(\cos(\theta_q/2)+\sin(\theta_q/2))\mid\rightarrow\rangle - \\ &\frac{1}{\sqrt{2}}(\cos(\theta_q/2)-\sin(\theta_q/2))\mid\leftarrow\rangle\ .\end{aligned} \tag{4.2}$$

Information-theoretic limits of QC discussed in [8] show that the probability $\mathcal{A}$ and $\mathcal{B}$ disagreement, if they choose different basis, is given by

$$q_{QM} = \frac{1}{2} - q_E = \frac{1}{2} - \frac{1}{4}\sin 2\theta_q. \tag{4.3}$$

For $\theta_q = 0$ the violation parameter q_E is zero and q_{QM} is one half. Thus if $\mathcal{A}$ and $\mathcal{B}$ compare colossal set of data M bits for which they have measured different basis, when quantum mechanics is not deformed, they find that the number of disagreement with expected $q_{QM} = \frac{1}{2}$ is zero with high accuracy . In this case QM is not deformed.

The probability $P(k)$ of k disagreements between $\mathcal{A}$ and $\mathcal{B}$ is given by the k-th term in the binomial expansion and for large M this distribution can be approximated by a Gaussian function, so that we find

$$P(k) \approx \frac{1}{\sqrt{2\Pi\sigma^2}} exp[-\frac{1}{2\sigma^2}[k - \frac{M}{2}(1-\theta_q)]^2] \tag{4.4}$$

where

$$\sigma^2 = \frac{M}{4}(1-{\theta_q}^2) \tag{4.5}$$

and the expected distribution in the absence of the violation is

$$P'(k) \approx \sqrt{2/M\Pi} exp[-\frac{2}{M}(k - M/2)^2] \ , \tag{4.6}$$

In our QC device we expect the accuracy of measurement $q_{QM} = \frac{1}{2}$ and $P'(k)$ approximately 0.001.

We shall further interpret this magnitude as g_E and show what it means for the minimal uncertainty in the q-deformed QM.

5 Q-deformed Heisenberg uncertainty relation in QC

We shall suppose that relation in operator algebra is not a commutator but the q-commutator, which can be generally written without indeces in the form

$$\mathbf{a}\,\mathbf{a}^+ - q^2\mathbf{a}^+\mathbf{a} = I \ . \tag{5.1}$$

We shall now discuss only the case $q > 1$.

In the obvious notation we express

$$a = \frac{1}{2}(\frac{x}{L} - \frac{p}{iI}) \ , \qquad a^+ = \frac{1}{2}(\frac{x}{L} + \frac{p}{iI}) \ , \tag{5.2}$$

where x and p are represented as symmetric operators with images that lies in their domain $D \subset$ Hilbert space. The constant L in front of the operator of coordinate and I in front of the operator of impulse carry units.

If we solve (5.2) using relation (5.1) we will get

$$\frac{1-q^2}{4}(\frac{x^2}{L^2} + \frac{p^2}{I^2}) + \frac{1+q^2}{4LI}[x,p] = 1 \tag{5.3}$$

We multiple (5.3) by $i\hbar$ and we obtain

$$[x,p] = i\hbar + i\hbar\frac{q^2-1}{4}\left(\frac{x^2}{L^2} + \frac{p^2}{I^2}\right) \ , \tag{5.4}$$

where the following relation between L and I is valid

$$I = \frac{\hbar}{4L}\left(q^2 + 1\right) . \tag{5.5}$$

and where $\hbar$ means Planck constant

$$\hbar = 6.625 \cdot 10^{-34} J.s. \tag{5.6}$$

For deformation parameter $q > 1$ in (5.1), the symmetric operators x and p in the domain D and assuming α real, there is valid analog of Weyl's proof:

$$|((x - \langle\psi, x\psi\rangle) + i\alpha(p - \langle\psi, p\psi\rangle))\psi| \geq)0 \ , \forall\psi \in D, \forall\alpha \tag{5.7}$$

and it can be written as

$$(\Delta x)^2 + \alpha^2(\Delta p)^2 + i\alpha\langle\psi, [x,p]\psi\rangle \geq 0, \tag{5.8}$$

where

$$(\Delta x)^2 = \langle\psi, (x - \langle\psi, x\psi\rangle)^2\psi\rangle . \tag{5.9}$$

So we get

$$(\Delta p)^2\left(\alpha - \frac{\hbar A}{2(\Delta p)^2}\right)^2 - \frac{h^2A^2}{4(\Delta p)^2} + (\Delta x)^2 \geq 0, \tag{5.10}$$

with

$$A = 1 + (q^2 - 1)\left(\frac{(\Delta x)^2 + \langle x\rangle^2}{4L^2} + \frac{(\Delta p)^2 + \langle p\rangle^2}{4I^2}\right) \tag{5.11}$$

The q-deformed Heisenberg uncertainty relation is

$$\Delta p \Delta x \geq \frac{\hbar}{2}A. \tag{5.12}$$

We assume that we measure the q-deformation $q_E \equiv q^2 - 1$.

For obtaining the minimal uncertainty we define

$$F(\Delta x, \Delta p) = \Delta x\Delta p - \frac{\hbar}{2}A, \tag{5.13}$$

and find $\triangle x_{min}$ by solving the following functional relations

$$\frac{\partial}{\partial \Delta p} F(\Delta x, \Delta p) = 0 \text{ and } F(\Delta x, \Delta p) = 0, \tag{5.14}$$

which has the solution

$$\Delta x_{min} = L^2 \frac{q_E}{q_E + 1}(1 + q_E(\frac{\langle x \rangle^2}{4L^2} + \frac{\langle p \rangle^2}{4L^2}) \ . \tag{5.15}$$

Thus the absolutely smallest position uncertainty which can be detected by the QC device is

$$\Delta x_0 = L\sqrt{\frac{q_E}{q_E + 1}} \tag{5.16}$$

and similar expression for the smallest uncertainty in the momentum may be derived.

As a conjecture, we suggest that the relation (5.12) can be written for the time and energy as is usual:

$$\Delta t \, \Delta E \geq \frac{\hbar}{2} \left[1 + f(q_0, (\Delta t)^2 + \langle t \rangle^2, (\Delta E)^2 + \langle E \rangle^2) \right] \tag{5.17}$$

with the minimal time-energy uncertainties

$$\Delta t_0 = M\sqrt{1 - q_0^2} \ , \qquad \Delta E_0 = N\sqrt{1 - q_0^2} \ ,$$

where the constants M and N carry units of time and energy; $MN = \frac{\hbar}{4}\,(q_0^2 + 1)$.

So we have the minimal uncertainty in the time and if we shall measure the two points on the time line by "infinitively" accurate apparatus, we have principal possibility to measure the points only with the minimal distance Δt_0. So we can define the minimal uncertainty in the time as the intertime interval and we look on the time as on the discrete variable.

Our model represents the new discrete time model from quantum physics.

If we assume $\mid L \mid = \mid I \mid \approx 10^{-17}$ and $q_E = 10^{-3}$ we get $\Delta x_0 = 6.10^{-19} m$. It is the limit for validity of ordinary QM in the space-time which is by orders higher then the value which can be obtained in high-energy physics. This is also the limit for the domain, where space-time is continuous, in the sense of our model.

6 Q-deformed quantum mechanics and quantum space-time

Limitations on the precision of localization in spacetime have appeared in the recent literature as consequence of different approaches to quantum gravity or q-deformed calculus [9].

We now show the coincidence between our q-deformed Heisenberg uncertainty relations, which can be verified by quantum cryptography, and a model of the discretization of spacetime.

Let us suppose that $q_E \equiv q^2 - 1 \approx 0$ is the parameter of the discretization of spacetime.

Let us consider the discretization of standard differential calculus in one space dimension

$$[x, dx] = dx q_E, \tag{6.1}$$

and the action of the discrete translation group

$$x^n dx = dx(x + q_E)^n, \tag{6.2}$$

$$\psi(x)dx = dx\psi(x + q_E), \tag{6.3}$$

for any wave function ψ of the Hilbert space of QM with the discrete space variable.

The discrete space variable is defined as $x = nq_E$, where n is an integer and and violation parameter q_E is the interval between two discrete space points in this space variable.

If we define the derivatives by

$$d\psi(x) = dx(\partial_x\psi)(x) = (\overleftarrow{\partial}\,\psi)(x)dx, \tag{6.4}$$

$$(\partial_x \psi)(x) = \frac{1}{q_E}[\psi(x+q_E) - \psi(x)], \tag{6.5}$$

$$(\overleftarrow{\partial_x} \psi)(x) = \frac{1}{q_E}[\psi(x) - \psi(x-q_E)], \tag{6.6}$$

$$(\overleftarrow{\partial_x} \psi)(x) = (\partial_x \psi)(x-q_E), \tag{6.7}$$

then the ordinary one-dimensional Schrödinger equation will be

$$\frac{1}{2}\frac{d^2\psi(x)}{dx^2} + [E - U(x)]\psi(x) = 0, \tag{6.8}$$

with the potential $U(x)$ and wavefunction $\psi(x) \equiv \psi(E, x)$, corresponding to energy value E, has on the discrete space the form

$$\frac{1}{2l^2}[\psi((n+1)q_E) - 2\psi(nq_E) + \psi((n-1)q_E)] + [E - U(nq_E)]\psi(nq_E) = 0. \tag{6.9}$$

We now show the coincidence between such discretization model, noncommutative differential calculus and q-deformed QM, assuming $q^2 \approx 1$.

Let us suppose that ordinary continuum space variable y in QM has the form:

$$y = \lim_{q_E \to 0}(1+q_E)^{\frac{x}{q_E}} = e^x. \tag{6.10}$$

Using Eqs.(6.4-6.7) and (6.10) we get:

$$\partial_y = y^{-1}\partial_x = (q_E + 1)^{\frac{-1}{q_E}}\partial_x \tag{6.11}$$

Thus, using $q_E \equiv q^2 - 1$, we have

$$(\partial_y \psi)(y) = \frac{\psi((q_E+1)y) - \psi(y)}{q_E y} = \frac{\psi(q^2 y) - \psi(y)}{(q^2-1)y} \tag{6.12}$$

$$(\overleftarrow{\partial_y} \psi)(y) = (q_E+1)\frac{\psi(y) - \psi((q_E+1)y))}{q_E y} = \frac{\psi(y) - \psi(q^2 y)}{(1-q^{-2})y} \tag{6.13}$$

what represents derivatives in the differential on the quantum hyperplane [9].

We can see that for $q_E = 0$ or $q^2 = 1$ we have the ordinary QM and continuous space-time.

In the application of this connection between the q-deformation and discretization on the time variable we can see that $t_1 t_2 = q t_2 t_1$ in q-deformed QM world. From this directly follows the arrow of the time. Of course in q-deformed QM also the T invariance is broken.

The T ivariance violation appears also in elementary particle physics. It is in the K_0 decay and parameter of the violation is in the absolute value 10^{-3}, what is very closed our value q_E.

7 Conclusions

An ideal opportunity for verifying the basic principles of quantum theory and possible q-deformation appears in the new discipline of physics and information theory—quantum cryptography.

It is thanks to the colossal statistical sets of data obtained from the QC device, which are exactly and accurately processed by the mathematical and measuring tests.

There are two possibilities of measuring the q-deformation on the quantum channel (i.e. optical fibre), which is the part of QC device:

1) the possibility of measuring the squeezed light and determining a squeezed deformation parameter q. It is not case of our interest

2) the possibility of the measuring q-deformed conjugate states in the interferometric scheme of quantum cryptography, what means the real measuring of the $|\uparrow\rangle, |\rightarrow\rangle$ states in original and deformed basis!

After excluding possibility the squeezed light states in QC device we have the possibility to verificate the q-deformation of Heisenberg uncertainty relation q-deformed QM and possible discretization on the base of presented model in Sec.4.

For a such strong physical conclusions theoretical-physical and technical analysis of the interferometric optical quantum cryptography device and experimental data must be done from the point of :

i) stability of measurement, false pulses and disturbances on optical system,

ii) the evaluation of experimental errors by methods of information theory.

In such a way optical quantum cryptography device is the cheapest experimental device for the verification of the validity of microworld laws.

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