RENDICONTI DEL CIRCOLO MATEMATICO DI PALERMO
Serie II, Suppl. 43 (1996), pp. 153-162

NATURAL SYMPLECTIC STRUCTURES ON THE TANGENT BUNDLE OF A SPACE–TIME

JOSEF JANYŠKA

ABSTRACT. All symplectic 2–forms naturally induced by the metric and a torsion free connection on the tangent bundle of a pseudo–Riemannian manifold are described. It is proved that the family of natural symplectic forms depends on a smooth real function and that all natural symplectic forms on the tangent bundle are pull–backs of the canonical symplectic form on the cotangent bundle with respect to diffeomorphisms naturally induced by the metric.

Introduction

In this paper we use the term "natural operator" in the sense of [5], [7], [11]. Namely, a natural operator is defined to be a system of local operators $A_M : C^\infty(FM) \to C^\infty(GM)$, such that $A_N(f^*_F s) = f^*_G A_M(s)$ for any section $(s : M \to FM) \in C^\infty(FM)$ and any (local) diffeomorphism $f : M \to N$, where F, G are two natural bundles, [8]. A natural operator is said to be of order r if, for all sections $s, q \in C^\infty(FM)$ and every point $x \in M$, the condition $j^r_x s = j^r_x q$ implies $A_M s(x) = A_M q(x)$. Then we have the induced natural transformation $\mathcal{A}_M : J^r FM \to GM$ such that $A_M(s) = \mathcal{A}_M(j^r s)$, for all $s \in C^\infty(FM)$. It is well known, that the correspondence between natural operators of order r and the induced natural transformations is bijective. In this paper by natural operators we mean the corresponding natural transformations. Briefly speaking, a natural operator is a fibred manifold mapping which is invariant with respect to local diffeomorphisms of the underlying manifold.

In general relativistic classical theories a distinguished role is played by symplectic 2-forms, which are induced from the space-time metric and connection. In fact, such forms can be used to describe basic structures of space-time itself and to formulate classical mechanics. Moreover, the quantization procedure, [10], [12], can be based

1991 *Mathematics Subject Classification.* Primary: 53C50, Secondary: 53C57, 53C80.
Key words and phrases. Symplectic 2–form, pseudo–Riemannian manifold, natural operator.
This paper is in final form and no version of it will be submitted for publication elsewhere.

on the choice of a symplectic 2-form on the background classical space-time and it is reasonable to take just one of the above 2-forms for this purpose. Of course, different choices would lead to different quantum theories; so, it is important to know whether the space-time metric and connection yield naturally a unique 2-form or not. In this paper we discuss the Einstein theory which is the most important general relativistic classical theory; it is based on a 4-dimensional manifold endowed with a Lorentz metric.

It is well known that a pseudo-Riemannian metric on M yields naturally a symplectic 2-form on the tangent space TM defined by $\Omega(g) = dd_v h$, where $h(u) = \frac{1}{2}g(u,u) \in C^\infty(TM)$ and d_v is the vertical differential, [1]. More generally, if a connection Γ on TM is given, then we can construct on TM a 2-form $\Omega(g,\Gamma)$ which is called the lift of g with respect to Γ to 2-forms on TM. These distinguished 2-forms are natural, but they are not the only natural ones. We describe all natural 2-forms on TM derived from g and a linear connection Γ and we express the conditions by which the 2-forms are closed and symplectic. In geometric quantization , [10], [12], a distinguished role is played by the canonical symplectic 2-form on the cotangent bundle of a manifold. We shall show that all natural symplectic 2-forms on the tangent bundle of a pseudo-Riemannian manifold are pull-backs of the canonical symplectic form on the cotangent bundle with respect to diffeomorphisms naturally induced by the metric.

In this paper M is a differentiable manifold with a pseudo-Riemannian metric g. Let (x^i) be a typical local chart on M, then (∂_i) and (d^i) denote the canonical local bases of modules of vector fields and forms on M. In classical general relativistic theories $\dim M = 4$ and g is a Lorentz metric, but it is not relevant for our purposes; our considerations are correct for non-orientable manifolds if $\dim M \geq 2$ and for orientable manifolds if $\dim M \geq 4$.

We consider the tangent bundle $\pi_M : TM \to M$ of M. The natural fibred coordinates are denoted by $(x^i, \dot{x}^i)$ and the canonical local bases of modules of vector fields and forms on TM by $(\partial_i, \dot{\partial}_i)$ and $(d^i, \dot{d}^i)$.

All manifolsds and mappings are assumed to be smooth.

1. Invariant functions on $\times^k TM$

1.1. Lemma. *Let us consider the standard tensor action of the group $SO(p, n-p, I\!R)$ on $I\!R^n$. Maximal dimension of orbits in $\times^k I\!R^n$ is*

i) $kn - \frac{k^2+k}{2}$ for $k \leq n-2$,

ii) $\frac{n^2-n}{2}$ for $k \geq n-1$.

Proof. Let $(u_a) \in \times^k I\!R^n$. The dimension of the stability subgroup of (u_a) is $\binom{n-d}{2}$, where d is number of linearly independent vectors in (u_a) and $d \leq (n-2)$. For $d > (n-2)$ the dimension of the stability group is equal to 0. The dimension of the orbit passing through (u_a) is

$$\dim SO(p, n-p, I\!R) - \dim(\text{stability subgroup of } u_a).$$

Hence in generic points, where the number of independent vectors in (u_a) is maximal, we get our Lemma 1.1. □

1.2. Lemma. *Let us consider on $\mathbb{R}^n$ the canonical pseudo-Riemannian metric*

$$g = \sum_{i=1}^{p} d^i \otimes d^i - \sum_{j=p+1}^{n} d^j \otimes d^j .$$

$SO(p, n-p, \mathbb{R})$-equivariant functions on $\times^k \mathbb{R}^n$ are of the form

$$f(g(u_a, u_b)), \qquad a, b = 1, \dots, k,\ a \le b,$$

where f is a function of $\frac{k(k+1)}{2}$ variables. Moreover, for $k \le n$, $g(u_a, u_b)$ form a functional base.

Proof. The Lie algebra of fundamental vector fields on $\mathbb{R}^n$, given by the action of $SO(p, n-p, \mathbb{R})$ on $\mathbb{R}^n$, is generated by vector fields

$$\zeta^i_j = x^i \frac{\partial}{\partial x^j} + x^j \frac{\partial}{\partial x^i}, \qquad i = 1, \dots, n,\ j = 1, \dots, p, i \ne j,$$

$$\zeta^i_j = x^i \frac{\partial}{\partial x^j} - x^j \frac{\partial}{\partial x^i}, \qquad i, j = p+1, \dots, n,\ i \ne j .$$

$SO(p, n-p, \mathbb{R})$-equivariant functions f on $\times^k \mathbb{R}^n$ are then solutions of the following homogeneous system of 1st order partial differential equations, [9],

$$\sum_{a=1}^{k} (\zeta^i_j)_a f = 0 . \tag{1.1}$$

By [9] a functional base of solutions of the system (1.1) has $(nk - d)$ functionally independent solutions, where d is maximal dimension of orbits of the action of $SO(p, n-p, \mathbb{R})$ on $\times^k \mathbb{R}^n$. If $k \le n$, then by Lemma 1.1 such a base consits of $\frac{k^2+k}{2}$ functionally independent solutions. For $k > n$ we have $nk - \frac{n^2-n}{2}$ functionally independent solutions. It is easy to see that $\frac{k^2+k}{2}$ functions

$$g(u_a, u_b) = \sum_{i=1}^{p} x^i_a x^i_b - \sum_{j=p+1}^{n} x^j_a x^j_b, \qquad a, b = 1, \dots, k,\ a \le b, \tag{1.2}$$

are solutions of the system (1.1). For $k \le n$ (1.2) constitutes a functional base of $SO(p, n-p, \mathbb{R})$-equivariant functions on $\times^k \mathbb{R}^n$. For $k > n$ (1.2) are functionally dependent and we can choose $nk - \frac{n^2-n}{2}$ functionally independent solutions which constitute a functional base of $SO(p, n-p, \mathbb{R})$-equivariant functions on $\times^k \mathbb{R}^n$. □

1.3. Example. Let us consider $n = 2$, $k = 3$ and the group $SO(2, \mathbb{R})$. Let us consider a generic element $(x, y, z) \in \times^3 \mathbb{R}^2$, $(x, y, z) \ne (0, 0, 0)$. By Lemma 1.2

$$g(x, x), g(y, y), g(z, z), g(x, y), g(x, z), g(y, z)$$

constitute a system of generators of $SO(2, \mathbb{R})$-equivariant functions on $\times^3 \mathbb{R}^2$ and every 5 functions of this system form a functional base. For instance, it is easy to see that

$$g(y, z) = \tag{1.3}$$

$$= \frac{g(x, y) g(x, z) \pm \sqrt{g(x, x) g(y, y) - (g(x, y))^2} \sqrt{g(x, x) g(z, z) - (g(x, z))^2}}{g(x, x)} .$$

1.4. As a direct consequence of Lemma 1.2 we have

Theorem. *Let (M, g) be a pseudo–Riemannian manifold. All invariant functions on $pRm(M) \times^k TM$ are of the form*

$$f(g(u_a, u_b)), \qquad a, b = 1, \dots, k,\ a \le b, \tag{1.4}$$

where f is a function of $\frac{k(k+1)}{2}$ variables.

Proof. This follows from the equivalence between invariant functions of the type $f(g, u_a)$ and functions of the type $f(u_a)$ invariant with respect to g–isomorphisms. The corresponding $SO(p, n-p, \mathbb{R})$–equivariant functions on the standard fibre $\times^k \mathbb{R}^n$ are described in Lemma 1.2. □

2. Natural F–metrics

2.1. Riemannian case. In what follows we shall use natural F–metrics which were defined by Kowalski and Sekizawa, [6], as natural operators from $pRm \times T$ to $T^* \otimes T^*$. Kowalski and Sekizawa completely classified all natural F-metrics for Riemannian metrics. We recall the original classification theorem by Kowalski and Sekizawa for sufficiently high dimensions.

Theorem. *Let (M, g) be an oriented Riemannian manifold of dimension $n \ge 4$. Then all natural F-metrics on M derived from g are symmetric and are of the form*

$$\beta_u(\xi, \eta) = \mu(||u||^2) g(\xi, \eta) + \nu(||u||^2) g(\xi, u) g(\eta, u) \tag{2.1}$$

$u \in T_x M$, where μ, ν are arbitrary functions defined on $\mathbb{R}^+ \cup \{0\}$. □

2.2. Pseudo–Riemannian case. Now we shall prove Theorem 2.1 for a pseudo–Riemannian manifold.

Theorem. *Let (M, g) be an oriented pseudo–Riemannian manifold of dimension ≥ 4. Then all natural F–metrics are of the form*

$$\beta_u(\xi, \eta) = \mu(h(u)) g_x(\xi, \eta) + \nu(h(u)) g_x(\xi, u) g_x(\eta, u), \tag{2.2}$$

$u \in T_x M$, where μ, ν are arbitrary real functions.

Proof. Natural F–metrics on M can be expressed as natural operators

$$pRm(M) \underset{M}{\times} TM \underset{M}{\times} TM \underset{M}{\times} TM \to M \times \mathbb{R}$$

which are linear in the last two summands. This is equivalent with natural operators from $TM \underset{M}{\times} TM \underset{M}{\times} TM$ to $M \times \mathbb{R}$ invariant with respect to g-isomorphisms. Let $f(u, \xi, \eta)$ is a corresponding $SO(p, n-p, \mathbb{R})$–equivariant function on the standard fibre. Then, by Lemma 1.2, f is a function of

$$g(u, u),\ g(u, \xi),\ g(u, \eta),\ g(\xi, \xi), g(\xi, \eta), g(\eta, \eta).$$

From linearity in ξ and η we get that unique bilinear functions are combinations of $g(\xi, \eta)$ and $g(u, \xi) g(u, \eta)$, where as coefficients stand real functions of $g(u, u)$. □

2.3. Remark. In Theorem 2.2 we have supposed $\dim M \geq 4$. For $\dim M \leq 3$ the result depends also on the signature of the metric. For instance, for $\dim M = 2$ and the signature (2,0), we get by Lemma 1.2 further, linear in ξ, solution of (1.1)

$$g(u, J\xi) = \sqrt{g(u,u)g(\xi,\xi) - (g(u,\xi))^2}\,, \tag{2.3}$$

where J is the canonical complex structure on $\mathbb{R}^2$. (2.3) and (1.4) implies

$$g(u, J\xi)g(u, J\eta) = g(u,\xi)g(u,\eta) - g(u,u)g(\xi,\eta).$$

Then all bilinear functions are combinations of

$$g(\xi,\eta)\,,\ g(u,\xi)g(u,\eta)\,,\ g(u,\xi)g(u,J\eta)\,,\ g(u,J\xi)g(u,\eta)\,,$$

where as coefficients stand functions of $g(u,u)$. By using the symmetrization and the antisymmetrization we obtain the classification theorem by Kowalski and Sekizawa in dimension 2.

3. Natural 2–forms on TM generated by g and Γ

3.1. Canonical symplectic form. The *canonical symplectic* 2-form on TM is defined by the formula

$$\Omega(g) = dd_v h \tag{3.1}$$

with coordinate expression

$$\Omega(g) = dd_v h = \partial_i g_{mj}\dot{x}^m d^i \wedge d^j - g_{ij} d^i \wedge \dot{d}^j\,. \tag{3.2}$$

In what follows we shall write $g_{jk,l}$ instead of $\partial_l g_{jk}$.

3.2. 2–form $\Omega(g,\Gamma)$. A linear connection Γ can be characterized by the TM-valued 1-form

$$\nu_\Gamma : TM \to T^*TM \underset{TM}{\otimes} TM$$

with coordinate expression

$$\nu_\Gamma = (\dot{d}^i - \Gamma^i_{jk}\dot{x}^k d^j) \otimes \partial_i\,. \tag{3.3}$$

The projection

$$T\pi_M : TTM \to TM$$

can be considered as the canonical TM-valued 1-form

$$\sigma : TM \to T^*TM \underset{TM}{\otimes} TM$$

with coordinate expression

$$\sigma = \delta^i_j d^j \otimes \partial_i\,. \tag{3.4}$$

The lift $\Omega(g,\Gamma)$ of the metric g with respect to a linear connection Γ to 2-forms on TM is then defined as

$$\Omega(g,\Gamma) = \nu_\Gamma \bar{\wedge} \sigma : TM \to \wedge^2 T^*TM, \tag{3.5}$$

where $\bar{\wedge}$ denotes the wedge product followed by the contraction through the metric g. In coordinates we have

$$\Omega(g,\Gamma) = g_{mj}(\dot{d}^m - \Gamma^m_{ik}\dot{x}^k\, d^i) \wedge d^j = g_{mi}\Gamma^m_{jk}\dot{x}^k\, d^i \wedge d^j - g_{ij}d^i \wedge \dot{d}^j\,. \tag{3.6}$$

3.3. Remark. Maximal rank of g implies that

$$\wedge^n \Omega$$

is a volume form on TM, i.e., that $\Omega(g,\Gamma)$ is non-degenerate.

3.4. Horizontal lift. The 2-form $\Omega(g,\Gamma)$ can be defined by another construction, [2]. Let us consider two vector fields ξ, η on M and denote by ξ^H the horizontal lift of ξ with respect to Γ and by ξ^V the vertical lift of ξ on TM. Then $\Omega(g,\Gamma)$ is characterized by

$$\begin{aligned} \Omega(g,\Gamma)(\xi^H,\eta^H) &= 0, \qquad \Omega(g,\Gamma)(\xi^H,\eta^V) = -g(\xi,\eta), \\ \Omega(g,\Gamma)(\xi^V,\eta^H) &= g(\eta,\xi), \qquad \Omega(g,\Gamma)(\xi^V,\eta^V) = 0. \end{aligned} \tag{3.7}$$

In [2] such construction is called the *horizontal lift* of g with respect to Γ to 2-forms on TM.

3.5. Remark. If Γ is the Levi-Civita connection on M, induced by the metric g, then it is easy to see that the coordinate expressions (3.2) and (3.6) coincide, i.e., the lift $\Omega(g,\Gamma)$ of g with respect to the Levi-Civita connection is the canonical symplectic form on TM.

4. Classification of natural 2–forms generated by g and Γ

4.1. From the viewpoint of natural geometry the canonical symplectic form described above is a natural operator from $J^1(pRm(M)) \times_M TM$ to $\wedge^2 T^*(TM)$ over the identity of TM. In [3] the following theorem was proved

Theorem. *Let (M,g) be an oriented pseudo-Riemannian manifold of dimension ≥ 4. Then all natural operators from $J^1(pRm(M)) \times_M TM$ to $\wedge^2 T^*(TM)$ over the identity of TM are horizontal lifts of natural F-metrics with respect to the Levi-Civita connection.* □

4.2. Now let us consider $C_\tau M \to M$ the bundle of torsion free linear connections on TM, then the construction of $\Omega(g,\Gamma)$ defined above is a natural operator from $pRm(M) \times_M C_\tau M \times_M TM$ to $\wedge^2 T^*(TM)$ over the identity of TM. By a simple modification of Theorem 4.1 we have

Theorem. *Let (M,g) be an oriented pseudo-Riemannian manifold of dimension ≥ 4. Then all natural operators from $pRm(M)) \times_M C_\tau M \times_M TM$ to $\wedge^2 T^*(TM)$ over the identity of TM are horizontal lifts of natural F-metrics with respect to connections.* □

4.3. Remark. The horizontal lift $\Omega(\beta,\Gamma)$ of a natural F-metric β with respect to a linear connection Γ to 2-forms on TM can be viewed in two different ways. The 2-form $\Omega(\beta,\Gamma)$ can be defined by (3.5), where $\bar{\wedge}$ is the wedge product followed by the contraction through β, or equivalently by (3.7), where β stands by g.

4.4. Remark. In Theorem 4.2 we have supposed Γ to be torsion free. This condition is essential. For connections with torsion we would obtain, instead of natural F-metrics, (0,2)–tensors which depend also on the torsion tensor of Γ.

4.5. We can give another geometrical description of $\Omega(\beta,\Gamma)$ which will be very useful in our considerations. Let $\phi : TM \to \wedge^r T^*TM \otimes_{TM} TM$ and $\psi : TM \to \wedge^s T^*TM \otimes_{TM} TM$ be two TM-valued forms on TM. Then a metric g on M (possibly also degenerate) admits two natural constructions of $(r+s)$-forms on TM.

For any vector fields $\Xi_1,\dots,\Xi_{r+s}$ on TM, $u \in TM$, we define $\phi\bar{\wedge}\psi : TM \to \wedge^{r+s}T^*TM$ by

$$(4.1)\quad (\psi\bar{\wedge}\psi)_u(\Xi_1,\dots,\Xi_{r+s}) = = \frac{1}{r!\,s!}\sum_{\sigma\in S_{r+s}} |\sigma|\, g(\phi(\Xi_{\sigma(1)},\dots,\Xi_{\sigma(r)}),\psi(\Xi_{\sigma(r+1)},\dots,\Xi_{\sigma(r+s)}))$$

and $\phi\bar{\bar{\wedge}}\psi : TM \to \wedge^{r+s}T^*TM$ by

$$(4.2)\quad (\phi\bar{\bar{\wedge}}\psi)_u(\Xi_1,\dots,\Xi_{r+s}) = = \frac{1}{r!\,s!}\sum_{\sigma\in S_{r+s}} |\sigma|\, g(\phi(\Xi_{\sigma(1)},\dots,\Xi_{\sigma(r)}),u)\, g(\psi(\Xi_{\sigma(r+1)},\dots,\Xi_{\sigma(r+s)}),u)\,.$$

Then (2.2) implies

$$(4.3)\qquad \Omega(\beta,\Gamma) = \mu(h(u))\nu_\Gamma\bar{\wedge}\sigma + \nu(h(u))\nu_\Gamma\bar{\bar{\wedge}}\sigma$$

and Theorem 4.2 can be reformulated as

Theorem. *All natural operators from $pRm(M)\times_M C_\tau M \times_M TM$ to $\wedge^2T^*(TM)$ over the identity of TM are of the form*

$$\mu(h(u))\nu_\Gamma\bar{\wedge}\sigma + \nu(h(u))\nu_\Gamma\bar{\bar{\wedge}}\sigma,$$

where μ,ν are arbitrary smooth functions of one real variable. □

5. Conditions for $\Omega(\beta,\Gamma)$ to be symplectic

5.1. Covariant exterior differential. Now we describe the conditions for $\Omega(\beta,\Gamma)$ to be symplectic for a general F-metric β given by (2.2). We shall use the covariant exterior differential d_Γ of vector–valued forms along morphisms with respect to a linear connection Γ, [5].

The metric g can be considered to be a T^*M-valued 1-form on M which will be denoted by $\bar{g}$ to distinguish it from the metric. Then, by [5],

$$(5.1)\qquad d_\Gamma\bar{g}(\xi,\eta)(\zeta) = \big(\nabla_\xi\bar{g}(\eta) - \nabla_\eta\bar{g}(\xi) - \bar{g}([\xi,\eta])\big)(\zeta)\,.$$

5.2. In [4] the following Theorem was proved.

Theorem. *Let Γ be a linear connection on M. Then $\Omega(\beta,\Gamma)$ is a symplectic form on TM if and only if*

$$\beta(g,u)(\xi,\eta)=\mu(h(u))g_x(\xi,\eta)+\frac{d\mu(h(u))}{dt}g_x(\xi,u)g_x(\eta,u), \tag{5.2}$$

$\xi,\eta,u\in T_xM$, where the real smooth function μ satisfies

$$\mu(t)\neq 0,\quad \mu(t)+2t\frac{d\mu(t)}{dt}\neq 0 \tag{5.3}$$

for all $t\in\mathbb{R}$. Moreover g and Γ have to satisfy

$$d_\Gamma\bar{g}=0\,, \tag{5.4}$$

$$g(\eta,u)\nabla_\xi g-g(\xi,u)\nabla_\eta g=0\,. \tag{5.5}$$

□

5.3. Geometric meaning of condition (5.4) is described in the following

Lemma. *Let Γ be a linear connection on M. The following three conditions are equivalent:*

i) $d_\Gamma\bar{g}=0$.

ii) *The forms $\Omega(g,\Gamma)$ and dd_vh coincide.*

iii) *$\Omega(g,\Gamma)$ is a symplectic form on TM.*

Proof. It is easy to prove this lemma by using coordinate expressions. By (5.1) the coordinate expression of $d_\Gamma\bar{g}=0$ is

$$g_{ki,j}-g_{kj,i}+g_{mi}\Gamma^m_{kj}-g_{mj}\Gamma^m_{ki}=0 \tag{5.6}$$

which is equivalent to the fact that the coordinate expressions (3.2) for dd_vh and (3.6) for $\Omega(g,\Gamma)$ coincide and from (3.6) it is also equivalent to $d\Omega(g,\Gamma)=0$. □

5.4. Remark. Let us remark that the condition $d_\Gamma\bar{g}=0$ does not imply Γ to be the Levi-Civita connection, but implies that the symplectic form generated by g and Γ coincides with the canonical symplectic form generated by g.

5.5. In Lemma 5.3 we have described the geometrical meaning of condition (5.4). Similarly for condition (5.5) we have

Lemma. *Let Γ be a linear connection on M. The following two conditions are equivalent:*

i) $g(\eta,u)\nabla_\xi g-g(\xi,u)\nabla_\eta g=0$ *for any $u\in TM$ and any vector fields ξ,η on M.*

ii) *The forms $dh\wedge d_vh$ and $\nu_\Gamma\bar{\bar{\Lambda}}\sigma$ coincide.*

Proof. Let ξ,η be two vector fields on M, ξ^V is the vertical lift and ξ^H is the horizontal lift (with respect to Γ) of ξ to TM. Then $(dh\wedge d_vh)_u(\xi^V,\eta^V)=0=(\nu_\Gamma\bar{\bar{\Lambda}}\sigma)_u(\xi^V,\eta^V)$ and $(dh\wedge d_vh)_u(\xi^H,\eta^V)=-g(\eta,u)g(\xi,u)=(\nu_\Gamma\bar{\bar{\Lambda}}\sigma)_u(\xi^H,\eta^V)$. Finally,

$$(dh\wedge d_vh)_u(\xi^H,\eta^H)=g(\eta,u)\nabla_\xi g(u,u)-g(\xi,u)\nabla_\eta g(u,u), \tag{5.7}$$

while $(\nu_\Gamma\bar{\bar{\Lambda}}\sigma)_u(\xi^H,\eta^H)=0$ which proves Lemma 5.5. □

5.6. Remark. If Γ is the Levi-Civita connection for g, then conditions (5.4) and (5.5) are satisfied in the canonical way. Hence the lift of a natural F-metric β with respect to the Levi-Civita connection to 2-forms on TM is a symplectic form if and only if β is given by (5.2), where (5.3) is satisfied. The most simple example of the function μ satisfying (5.3) is $\mu(t) = C \neq 0$. A non trivial example is for instance

$$\mu(t) = t^2 + c, \qquad 0 < c \in I\!R.$$

Hence the pseudo-Riemannian metric on M induces naturally on TM a family of symplectic forms and this family depends on one smooth real function satisfying (5.3).

5.7. Theorem. *Let Γ be a linear connection on M and μ a real function such that*

$$\Omega(\beta, \Gamma) = \mu(h(u))\nu_\Gamma\bar{\Lambda}\sigma + \frac{d\mu(h(u))}{dt}\nu_\Gamma\bar{\bar{\Lambda}}\sigma \tag{5.8}$$

is a symplectic form, then $\Omega(\beta, \Gamma) = d\alpha$, where $\alpha = \mu(h(u))d_v h$.

Proof. Theorem 5.2 implies that $\Omega(\beta, \Gamma)$ is a symplectic form if and only if μ satisfies (5.3) and g is related with Γ by conditions (5.4) and (5.5). (5.4) and Lemma 5.3 imply $\nu_\Gamma\bar{\Lambda}\sigma \equiv dd_v h$ and (5.5) and Lemma 5.5 imply $\nu_\Gamma\bar{\bar{\Lambda}}\sigma \equiv dh \wedge d_v h$ which prove Theorem 5.7. □

5.8. Remark. Theorem 5.3 and Theorem 5.7 imply that any symplectic form on TM naturally induced by g and Γ is in fact independent of Γ and coincides with a symplectic form naturally induced by g only (by g and the Levi-Civita connection for g). Hence all natural symplectic forms on TM generated by g and Γ are decsribed in Remark 5.6. From Theorem 5.7 it follows that for any natural symplectic form there exists a natural (global) symplectic potential α, namely $\Omega = d\alpha$, where $\alpha = \mu(h(u))d_v h$.

6. Relations with T^*M

In this section we describe the relations between natural symplectic structures on TM described above and the canonical symplectic structure on T^*M.

6.1. Canonical symplectic form on T^*M. Let $q_M : T^*M \to M$ be the cotangent bundle of the space–time (M, g). On T^*M we have the canonical (natural) symplectic 2–form ω given by the exterior differential of the Liouville 1–form θ.

In the natural coordinate chart (x^i, x_i) on T^*M we have

$$\theta = x_i d^i, \qquad \omega = d_i \wedge d^i . \tag{6.1}$$

6.2. The metric g induce the canonical fibred diffeomorphism $g^\flat : TM \to T^*M$ over M by $g^\flat(u) = g(\cdot, u)$. In coordinates $(g^\flat(u))_i = g_{ij}u^i$.

Then we have

$$d_v h = (g^\flat)^*\theta \tag{6.2}$$

and

$$\Omega(g) = dd_v h = d((g^\flat)^*\theta) = (g^\flat)^*\omega . \tag{6.3}$$

6.3. Lemma. *All fibred manifold morphisms $TM \to T^*M$ over M naturally induced by the metric g are of the form*

$$\mu(h(u))g^\flat(u)\,, \tag{6.4}$$

where μ is a function on $\mathbb{R}$.

Proof. It is a direct consequence of Theorem 1.4. □

6.4. Now we can obtain the relations between the canonical symplectic 2-form on T^*M and all natural symplectic 2-forms on TM.

Theorem. *Any natural symplectic 2-form on TM naturally induced by g and a torsion free linear connection is obtained by the pull-back of the canonical symplectic 2-form on T^*M with respect the fibred morphism $\mu g^\flat$, where a real function μ satisfies the conditions*

$$\mu(t) \neq 0, \quad \mu(t) + 2t\frac{d\mu(t)}{dt} \neq 0$$

for any $t \in \mathbb{R}$.

Proof. The pull-back of the canonical symplectic form with respect to the morphism (6.3) is the form (5.8) which is a symplectic form on TM if and inly if (5.3) is satisfied. Theorem 5.7 then implies Theorem 6.4. □

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Department of Mathematics, University of Masaryk
Janáčkovo nám 2a, 662 95 Brno, CZECH REPUBLIC
E-mail address: janyska@math.muni.cz

RENDICONTI DEL CIRCOLO MATEMATICO DI PALERMO
Serie II, Suppl. 43 (1996), pp. 163-171

CONNECTIONS IN FIRST PRINCIPAL PROLONGATIONS

Ivan Kolář, George Virsik

Abstract. We introduce the conjugate connection of a connection Δ in the first principal prolongation W^1P of a principal bundle $P \to M$, and study the torsion of Δ. Next we determine all gauge-natural operators transforming a connection in P and a linear symmetric connection on M into a connection in W^1P. Finally we relate these results to general connections induced in the first jet prolongation of a fiber bundle associated to P.

Keywords. Conjugate connection, torsion, first principal prolongation, first jet prolongation, gauge-natural operator **AMS Classification.** 53 C 05, 58 A 20

The basic ideas of the theory of principal prolongations of principal bundles were introduced by C. Ehresmann in groupoid form, [3]. A crucial part of this theory is the fact that the r-th order jet prolongation J^rE of a fiber bundle E associated to a principal bundle P is associated to the r-th principal prolongation W^rP of P, [12], [14], [17]. The fundamental role of W^rP in the theory of gauge-natural bundles and operators was observed by D. J. Eck, [2], and further studied in [12]. Moreover, there is a canonical inclusion $P^rM \subset W^1(P^{r-1}M)$ of the r-th order frame bundle P^rM of a manifold M into the first principal prolongation of the $(r-1)$-st order frame bundle of M. In this sense W^1P is a useful recurrence model for higher order differential geometry, see [8], [9], [16].

In the introductory section of the paper we summarize some basic facts about W^1P and connections in W^1P. Note that connections in W^1P correspond to second order total connections in P in the language of [19]. Next the conjugate connection of a connection Δ in W^1P is introduced and the torsion of Δ is studied systematically. In Section 4 we determine all gauge-natural operators transforming a connection in P and a linear symmetric connection on the base manifold into a connection in W^1P. Our list clarifies some further aspects of the geometry of connections in W^1P. The last section discusses some basic properties of general connections induced in associated fiber bundles.

All manifolds and maps are assumed to be infinitely differentiable.

This paper is in final form and no version of it will be submitted for publication elsewhere.

The first author was supported by the grant 201/93/2125 of the GA ČR. He also gratefully acknowledges financial support he received from Monash University, Australia, during his stay at their Department of Mathematics while this paper was in preparation.

1. Preliminaries. Given a principal bundle $P(M,G)$ with structure group G, the connection bundle $QP \to M$ of P can be defined as a factor bundle $QP = J^1P/G$, where J^1P is the first jet prolongation of P. This is an affine bundle with the modelling vector bundle $LP \otimes T^*M$, LP denoting the adjoint bundle, i.e. the vector bundle associated to P with respect to the adjoint action of G on its Lie algebra $\mathfrak{g}$, [12]. A connection Γ in P is defined either as a right-invariant section $\Gamma : P \to J^1P$ or as a section $\Gamma : M \to QP$. The difference of two connections on P is a tensor field of type $LP \otimes T^*M$.

Write $m = \dim M$. The first principal prolongation W^1P of P is the space of all 1-jets at $(0,e) \in \mathbb{R}^m \times G$, of local principal bundle isomorphisms $\mathbb{R}^m \times G \to P$, where e is the unit of G. It follows that W^1P is a principal bundle over M, the structure group of which is $W^1_mG = W^1_0(\mathbb{R}^m \times G)$, where both the multiplication in W^1_mG and the right action of W^1_mG on W^1P are defined by jet composition. In the special case when $G = \{e\}$, we obtain the first order frame bundle $P^1M(M, G^1_m)$ of M. One finds easily that the fibered manifold $W^1P \to M$ coincides with the fibered product $W^1P = P^1M \times_M J^1P$ over M, [12]. If $T^1_mG = J^1_0(\mathbb{R}^m, G)$ is the group of all $(1,m)$-velocities on G, then W^1_mG coincides with the semidirect product $G^1_m \rtimes T^1_mG$ with respect to the jet action of G^1_m on T^1_mG. We have two canonical principal bundle morphisms $p_1 : W^1P \to P^1M$ and $p_2 : W^1P \to P$. In particular, every connection Δ in W^1P gives rise to underlying connections $p_1\Delta$ in P^1M and $p_2\Delta$ in P.

We recall a construction transforming every pair of connections $\Gamma : P \to J^1P$ and $\Lambda : P^1M \to J^1P^1M$ into a connection $p(\Gamma,\Lambda)$ in W^1P, [7]. Define

$$R(\Gamma) = \{(u, \Gamma(v)), (u,v) \in P^1M \times_M P\}. \tag{1}$$

We have an injection $i : G \to T^1_mG$, $g \mapsto j^1_0\hat{g}$, where $\hat{g}$ stands for the constant map of $\mathbb{R}^m$ into $g \in G$. One finds easily that $R(\Gamma)$ is a reduction of W^1P to $G^1_m \times i(G) \subset W^1_mG$, [8]. Hence (1) identifies $R(\Gamma)$ with $P^1M \times_M P$. Thus the product connection $\Lambda \times \Gamma$ on $P^1M \times_M P$ is identified with a connection in $R(\Gamma)$ and the latter connection is uniquely extended into a connection $p(\Gamma,\Lambda)$ in W^1P (see [19] for the groupoid form of this construction). Clearly, $p_1(p(\Gamma,\Lambda)) = \Lambda$ and $p_2(p(\Gamma,\Lambda)) = \Gamma$.

The projections p_1 and p_2 give rise to projections $LW^1P \otimes T^*M \to LP^1M \otimes T^*M = TM \otimes \overset{2}{\otimes} T^*M$ and $LW^1P \otimes T^*M \to LP \otimes T^*M$. The common kernel of these projections is $LP \otimes \overset{2}{\otimes} T^*M$, [10]. Write $C^\infty Y$ for the set of all sections of a fibered manifold $Y \to M$.

Proposition 1. *Connections in W^1P are in bijection with triples (Γ,Λ,D), where $\Gamma \in C^\infty(QP)$, $\Lambda \in C^\infty(QP^1M)$ and $D \in C^\infty(LP \otimes \overset{2}{\otimes} T^*M)$.*

Proof. For $\Delta \in C^\infty(QW^1P)$ set $\Gamma = p_2\Delta$, $\Lambda = p_1\Lambda$ and $D = \Delta - p(p_2\Delta, p_1\Delta)$. □

The factor $T^*M \otimes T^*M$ gives rise to an exchange map ex : $LP \otimes \overset{2}{\otimes} T^*M \to LP \otimes \overset{2}{\otimes} T^*M$. Thus, if we replace Λ by the classical conjugate connection $\widetilde{\Lambda}$ and D by ex $\circ\, D$, we obtain an involutive operation on connections in W^1P. The next section deals with this involution in more detail.

2. Conjugation of connections in W^1P. Consider a connection Δ in W^1P in the form $\Delta : W^1P \to J^1(W^1P) = J^1P^1M \times_M J^1J^1P$. Let $\beta : J^1P \to P$ be the target jet projection. On $J^1J^1P = J^1(J^1P \to M)$ we have both the target jet projection $\beta_1 : J^1J^1P \to J^1P$ and the first jet prolongation $J^1\beta : J^1J^1P \to J^1P$ of β. The subset $\bar{J}^2P \subset J^1J^1P$ of all elements U satisfying $\beta_1(U) = J^1\beta(U)$ is called the second semiholonomic prolongation of P. There is an induced action of G on $\bar{J}^2P$ and the factor bundle $\bar{Q}^2P = \bar{J}^2P/G$ is said to be the bundle of second order semiholonomic connections in P, [5]. This is an affine bundle over QP with the modelling vector bundle $LP \otimes \overset{2}{\otimes} T^*M$. A section Σ of $\bar{Q}^2P \to M$ is called a second order semiholonomic connection in P. Equivalently, Σ can be considered as a right-invariant section $P \to \bar{J}^2P$.

Let $\Gamma = p_2\Delta$. For $(u,v) \in P^1M \times_M P$ we can construct the second projection

$$\mathrm{pr}_2\Delta(u, \Gamma(v)) \in J^1J^1P. \tag{2}$$

By [8], (2) lies in $\bar{J}^2P$ and is independent of u. Hence (2) defines a map $\mu(\Delta) : P \to \bar{J}^2P$, which is a second order semiholonomic connection. Conversely, if $\Sigma : P \to \bar{J}^2P$ is a second order semiholonomic connection in P over $\Gamma : P \to J^1P$ and $\Lambda : P^1M \to J^1P^1M$ is a connection in P^1M, then the rule

$$(u,v) \mapsto (\Lambda(u), \Sigma(v)) \tag{3}$$

defines a right-invariant section $R(\Gamma) \to J^1W^1P$, which is uniquely extended into a connection (Σ, Λ) in W^1P. This proves

Proposition 2. *The map $\Delta \mapsto (\mu(\Delta), p_1\Delta)$ establishes a bijection between connections in W^1P and pairs consisting of a second order semiholonomic connection in P and a classical linear connection on M.*

The groupoid form of this result is Proposition 1 of [19].

J. Pradines introduced a canonical involution κ of semiholonomic 2-jets, [15]. In particular, $\kappa_P : \bar{J}^2P \to \bar{J}^2P$ is a fibered manifold morphism over the identity of J^1P. We recall that $\widetilde{\Lambda}$ denotes the classical conjugate connection to $\Lambda \in C^\infty QP^1M$.

Definition 1. If $\Delta = (\Sigma, \Lambda)$ is a connection in W^1P, then $\widetilde{\Delta} = (\kappa_P \circ \Sigma, \widetilde{\Lambda})$ is called the conjugate connection of Δ.

Note that we could have made a more subtle distinction: If $\Delta = (\Sigma, \Lambda)$, then $(\kappa_P \circ \Sigma, \Lambda)$ can be said to be the partially 1-conjugate connection of Δ, while $(\Sigma, \widetilde{\Lambda})$ can be called the partially 2-conjugate connection of Δ.

Next we shall show that this definition of $\widetilde{\Delta}$ coincides with the one outlined at the end of Section 1. For every connection $\Gamma : P \to J^1P$, we can construct the jet prolongation $J^1\Gamma : J^1P \to J^1J^1P$. The values of $\Gamma' = J^1\Gamma \circ \Gamma$ lie in $\bar{J}^2P$ and Γ' is a second order semiholonomic connection in P, which we call the Ehresmann prolongation of Γ, [4]. Since $\bar{Q}^2P \to QP$ is an affine bundle, every two second order semiholonomic connections Σ_1, Σ_2 in P over the same connection in P define a section $\Sigma_1 - \Sigma_2$ of $LP \otimes \overset{2}{\otimes} T^*M$. Hence every $\Sigma \in C^\infty(\bar{Q}^2P)$ over $\Gamma \in C^\infty(QP)$ can be written as $\Sigma = \Gamma' + d$ with $d = \Sigma - \Gamma' \in C^\infty(LP \otimes \overset{2}{\otimes} T^*M)$. This establishes a bijection (cf. also Theorem 10 of [18]).

$$\Sigma = (\Gamma, d), \quad \Sigma \in C^\infty \bar{Q}^2 P, \quad \Gamma \in C^\infty QP, \quad d \in C^\infty(LP \otimes \overset{2}{\otimes} T^*M). \tag{4}$$

By [6], if U, $V \in \bar{J}^2 P$ are over the same element of $J^1 P$, then $\kappa_P(U) - \kappa_P(V) = \mathrm{ex}(V - U) \in LP \otimes \overset{2}{\otimes} T^*M$. It follows from the construction of $p(\Gamma, \Lambda)$, that $\Delta = (\Sigma, \Lambda)$ and $\Sigma = (\Gamma, d)$ imply $\Delta = (\Gamma, \Lambda, d)$ in the sense of Proposition 1. This proves

Proposition 3. *If* $\Delta = (\Gamma, \Lambda, D)$, *then* $\widetilde{\Delta} = (\Gamma, \widetilde{\Lambda}, \mathrm{ex} \circ D)$.

3. The torsion. The torsion of a connection in $W^1 P$ was introduced in [8] as follows. There is a canonical $(\mathbb{R}^m \times \mathfrak{g})$-valued 1-form Θ on $W^1 P$, which generalizes the classical soldering form $TP^1 M \to \mathbb{R}^m$ of $P^1 M$. This is a pseudotensorial 1-form with respect to an action l of $W^1_m G$ on $\mathbb{R}^m \times \mathfrak{g}$. This action is completely described in [12], p. 155. Here we need only the fact that the restriction l_0 of l to $G^1_m \times i(G) \subset W^1_m G$ is the sum of the standard action of G^1_m on $\mathbb{R}^m$ and of the adjoint action of G on $\mathfrak{g}$.

Definition 2. The torsion of a connection Δ on $W^1 P$ is the covariant exterior differential $d_\Delta \Theta$.

Consider the associated vector bundle $E = W^1 P[\mathbb{R}^m \times g, l]$. By Proposition 11.14 of [12], $d_\Delta \Theta$ can be interpreted as an E-valued 2-form on M. If we replace $W^1 P$ by its reduction $R(\Gamma)$, $\Gamma = p_2 \Delta$, then $R(\Gamma)[\mathbb{R}^m \times \mathfrak{g}, l_0]$-valued 2-forms on M are identified with sections of $(TM \times_M LP) \otimes \wedge^2 T^*M$. In this sense we shall write $d_\Delta \Theta : M \to (TM \times_M LP) \otimes \wedge^2 T^*M$.

On the other hand, the torsion of a classical connection Λ in $P^1 M$ is $\Lambda - \widetilde{\Lambda}$. To deduce a similar result for connections in $W^1 P$, we are going to express $d_\Delta \Theta$ in terms of the so-called difference tensor of a semiholonomic 2-jet. By definition, the restriction Δ_R of Δ to $R(\Gamma) = P^1 M \times_M P$ is a map $\Delta_R : P^1 M \times_M P \to J^1 P^1 M \times_M \bar{J}^2 P$. There is a canonical identification $J^1 P^1 M \approx \bar{P}^2 M$, where $\bar{P}^2 M$ is the space of all second order semiholonomic frames on M, [13]. Hence we can write $\Delta_R : P^1 M \times_M P \to \bar{P}^2 M \times_M \bar{J}^2 P$. Note that this map was referred to as the total reduction of Δ in the groupoid languages of [19], Proposition 2.

In general, the difference tensor $\delta(U)$ of a semiholonomic 2-jet U is the difference $U - \kappa(U)$, [6], [15]. In our case, the composition $\delta \circ \Delta_R$ is identified with a section $M \to (TM \times_M LP) \otimes \wedge^2 T^*M$. By Proposition 5 from [9] we have

$$d_\Delta \Theta = \delta \circ \Delta_R . \tag{5}$$

The difference of two arbitrary connections in $W^1 P$ is a section of $LW^1 P \otimes T^*M$. In general, $LW^1 P$ is a fiber bundle associated to $W^1 P$. In the case of Δ and $\widetilde{\Delta}$ we replace $W^1 P$ by $R(\Gamma)$ as in our approach to $d_\Delta \Theta$. Then comparison of (5) with Definition 1 yields

Proposition 4. *The torsion of* Δ *coincides with* $\Delta - \widetilde{\Delta}$.

The second principal prolongation W^2P of P equals $P^2M \times_M J^2P$. This defines an inclusion $W^2P \subset J^1P^1M \times_M \bar{J}^2P$. The result below, obtained in [1] by means of a more involved approach via structure equations, follows then directly from (5).

Corollary 1. $d_\Delta \Theta = 0$ *iff the values of* Δ_R *lie in* W^2P.

Thus Δ_R is torsion-free iff, in the language of [19], it has a holonomic total reduction. A similar result for connections in the r-th order frame bundle P^rM can be found in [11].

4. Gauge-natural operators. The theory of gauge-natural bundles and gauge-natural operators was established by D. J. Eck in [2]. Gauge-natural operators between two gauge-natural bundles can be identified with the corresponding geometric operators [12]. The $p(\Gamma, \Lambda)$ constructed above is such a gauge-natural operator transforming a connection in P and a connection in P^1M into a connection in W^1P. So we find it instructive to determine all gauge-natural operators of this type. For the particular case of $G = GL(n, \mathbb{R})$ this problem was solved in [10].

Proposition 6 of [10] demonstrates that the torsion of Λ gives rise to several operators, which are of secondary geometric importance. Therefore we shall restrict ourselves to the case when Λ is a symmetric connection. Let $Q_\tau P^1M$ denote the bundle of the linear symmetric connections on M, [12].

Since the difference of two connections in W^1P is a section of $LW^1P \otimes T^*M$ and we know one operator $p(\Gamma, \Lambda)$, our problem is to find all gauge-natural tensor fields of type $LW^1P \otimes T^*M$. We shall see that the values of all of them lie in the subbundle $LP \otimes \overset{2}{\otimes} T^*M$. So let us start with the description of some simple operators of this type. The curvature $C(\Gamma)$ of Γ is an (antisymmetric) section of $LP \otimes \overset{2}{\otimes} T^*M$. Let $Z \subset L(\mathfrak{g}, \mathfrak{g})$ be the subspace of all linear maps commuting with the adjoint action of G. Since every $z \in Z$ is an equivariant map between the standard fibers, it induces a vector bundle morphism $z_P : LP \to LP$. Hence one can construct a modified curvature operator $C(\Gamma)(z) = (z_P \otimes id) \circ C(\Gamma)$, [12]. On the other hand, by Example 28.7 of [12], all natural operators $C^\infty(Q_\tau P^1M) \to C^\infty(T^*M \otimes T^*M)$ are linearly generated by two contractions $R_1(\Lambda) = (R^k_{kij})$ and $R_2(\Lambda) = (R^k_{ikj})$ of the curvature tensor (R^i_{jkl}) of Λ. Let $S \subset \mathfrak{g}$ be the subspace of all vectors invariant with respect to the adjoint action. Since $A \in S$ is an invariant element of the standard fiber, it determines a section A_P of LP.

Proposition 5. *All gauge-natural operators from* $C^\infty(QP) \times C^\infty(Q_\tau P^1M)$ *to* $C^\infty(QW^1P)$ *are of the form*

$$p(\Gamma, \Lambda) + C(\Gamma)(z) + A_P \otimes R_1(\Lambda) + B_P \otimes R_2(\Lambda) \tag{6}$$

for all $z \in Z$ *and all* $A, B \in S$.

Proof. This is heavily based on procedures developed in [12]. By Propositions 23.5 and 51.16 of [12], every gauge-natural operator D of our type has finite order. Fix a basis e_p of $\mathfrak{g}$. Let $a \in G$, and let a^i_j, a^p_i be the coordinates in W^1_mG in the sense of [12],

p. 399, and let (a^i_j, a^i_{jk}) be the standard coordinates in the second order jet group G^2_m of dimension m. Denote by Γ^p_i the corresponding coordinates in the standard fiber of QP and by $\Lambda^i_{jk} = \Lambda^i_{kj}$ the canonical coordinates in the standard fiber of $Q_\tau P^1M$. Let D^i_{jk}, D^p_i, D^p_{jk} be the coordinate components of D. First one deduces that $p_1D(\Gamma,\Lambda) = \Lambda$. By the homogeneous function theorem ([12], p. 213), D^i_{jk} is linear in both Λ^i_{jk} and Γ^p_i. If one applies the invariant tensor theorem ([12], p. 214), to the part linear in Γ^p_i, one obtains

$$D^i_{jk} = b_p\delta^i_j\Gamma^p_k + c_p\delta^i_k\Gamma^p_j, \qquad b_p,\ c_p \in \mathbb{R}.$$

Equivariance on $\mathfrak{g} \otimes \mathbb{R}^m \subset W^1_mG$ yields $b_p\delta^i_ja^p_k + c_p\delta^i_ka^p_j = 0$, which implies $b_p = 0 = c_p$. Then $D^i_{jk} = \Lambda^i_{jk}$ by Remark 25.3 of [12].

Analogously one finds $p_2D(\Gamma,\Lambda) = \Gamma$. Indeed, by the homogeneous function theorem, D^p_i is linear in both Λ^i_{jk} and Γ^p_i. Let us consider the part linear in Λ^i_{jk}. By the invariant tensor theorem we have $D^p_i = b^p\Lambda^j_{ji}$. Equivariance with respect to G^2_m yields $b^pa^j_{ji} = 0$, i.e. $b^p = 0$. Thus for a fixed p we have $D^p_i = b\Gamma^p_i$ and equivariance on $\mathfrak{g}\otimes\mathbb{R}^m$ yields $b\Gamma^p_i + a^p_i = b(\Gamma^p_i + a^p_i)$. This implies $b = 1$.

Hence the difference $D(\Gamma,\Lambda) - p(\Gamma,\Lambda)$ is a gauge-natural section of $LP \otimes \overset{2}{\otimes} T^*M$. Let Γ^p_{ij}, or Λ^i_{jkl} be the additional coordinates on the standard fiber of J^1QP or $J^1Q_\tau P^1M$, respectively. By the homogeneous function theorem, D^p_{ij} is quadratic in Λ^i_{jk}, linear in Λ^i_{jkl}, quadratic in Γ^p_i, linear in Γ^p_{ij} and bilinear in Λ^i_{jk} and Γ^p_i. Equivariance on $\mathfrak{g}\otimes\mathbb{R}^m$ and on the kernel of the jet projection $G^2_m \to G^1_m$ implies that the expression bilinear in Λ^i_{jk} and Γ^p_i vanishes. By Example 28.7 of [12], the terms in Λ^i_{jk}, Λ^i_{jkl} are of the form

$$D^p_{ij} = b^pR_1 + c^pR_2, \qquad b^p,\ c^p \in \mathbb{R}.$$

Equivariance on $G \subset W^1_mG$ implies that both (b^p) and (c^p) are Ad-invariant elements of $\mathfrak{g}$. The part in Γ^p_i, Γ^p_{ij} corresponds to a gauge-natural operator of the curvature type. By Proposition 52.5 of [12], all such operators are the modified curvature operators. □

5. The flow prolongation. We present another simple construction transforming a connection Γ in P and a connection Λ in P^1M into a connection $\mathcal{W}^1(\Gamma,\Lambda)$ in W^1P. It is based on a general idea from Section 45 of [12]. Let X be a vector field on M and let ΓX be its Γ-lift on P. Since W^1 is a functor on the category of G-bundles with m-dimensional bases and local isomorphisms, we have the situation of 45.3 in [12]. In particular, the flow prolongation $\mathcal{W}^1(\Gamma X)$ of the vector field ΓX is a vector field on W^1P, and it gives rise to a bundle map $\mathcal{W}^1(\Gamma) : W^1P \times_M J^1TM \to TW^1P$. The connection Λ can be interpreted as a map $\Lambda : TM \to J^1TM$. By Proposition 45.6 of [12], $\mathcal{W}^1(\Gamma)\circ(id\times_M\Lambda) : W^1P\times_M TM \to TW^1P$ is the lifting map of a connection $\mathcal{W}^1(\Gamma,\Lambda)$ in W^1P.

If Λ is symmetric, then $\mathcal{W}^1(\Gamma,\Lambda)$ belongs to the list (6). Hence the difference $p(\Gamma,\Lambda) - \mathcal{W}^1(\Gamma,\Lambda)$ must be one of the tensor fields from (6). Since our construction is independent of the structure group, Proposition 5 suggests that it is a constant multiple of $C(\Gamma)$. A further analysis, which involves the case of arbitrary Λ, leads to the following assertion, the proof of which will be given at the end of Section 7, since it is based on some additional geometric results.

Proposition 6. $p(\Gamma, \Lambda) - \mathcal{W}^1(\Gamma, \widetilde{\Lambda}) = C(\Gamma)$.

6. Induced connections. In an arbitrary fibered manifold $Y \to M$ a general connection is defined as a section $\Gamma : Y \to J^1Y$. In particular, if $Y = P(M, G)$ is a principal bundle and Γ is right-invariant, we refer to Γ as a principal connection.

Let F be a left G-space and $P[F]$ be the fiber bundle associated to P with standard fiber F. Every principal connection Γ in P induces a general connection $\Gamma[F]$ in $P[F]$ as follows. Every element $y \in P[F]$ is an equivalence class $y = \{v, a\}$, $v \in P$, $a \in F$. It $\Gamma(v) = j^1_x\sigma$, where σ is a local section of P, put

$$\Gamma[F](y) = j^1_x\{\sigma, a\}\,. \tag{7}$$

$\Gamma[F$ is well defined since Γ is right-invariant.

A general connection Γ in $Y \to M$ together with a linear connection $\Lambda : TM \to J^1TM$ induce a connection $\mathcal{J}^1(\Gamma, \Lambda)$ in $J^1Y \to M$ analogously to Section 5 as follows. Every vector field X on M is lifted into a vector field ΓX on Y. The flow prolongation $\mathcal{J}^1(\Gamma X)$ of such a vector field gives rise to a map $\mathcal{J}^1(\Gamma) : J^1Y \times_M J^1TM \to TJ^1Y$. Then $\mathcal{J}^1(\Gamma) \circ (\mathrm{id} \times_M \Lambda) : J^1Y \times_M TM \to TJ^1Y$ is the lifting map of a general connection $\mathcal{J}^1(\Gamma, \Lambda)$ in $J^1Y \to M$.

If $E = P[F]$ is a fiber bundle associated to P, then its first jet prolongation is a fiber bundle associated to W^1P with standard fiber $T^1_mF = J^1_0(\mathbb{R}^m, F)$ (see [12], p. 152). Hence every principal connection Δ in W^1P induces a connection $\Delta[T^1_mF]$ in J^1E. In particular, if Γ is a principal connection in P, then $\mathcal{W}^1(\Gamma, \Lambda)[T^1_mF]$ is a general connection in J^1E. On the other band, also $\mathcal{J}^1(\Gamma[F], \Lambda)$ is a general connection in J^1E.

Proposition 7. $\mathcal{W}^1(\Gamma, \Lambda)[T^1_mF] = \mathcal{J}^1(\Gamma[F], \Lambda)$.

Proof. Every $a \in F$ defines a map $\varphi_a : P \to E, v \mapsto \{v, a\}$. Clearly, definition (7) of the induced connection is equivalent to the requirement that the vector fields ΓX and $\Gamma[F](X)$ be φ_a-related for all $a \in F$ and all $X \in C^\infty TM$. By Proposition 15.5 of [12], vector fields $\mathcal{W}^1(\Gamma X)$ and $\mathcal{J}^1(\Gamma[F](X))$ are φ_A-related for all $A \in T^1_mF$. Therefore it suffices to prove that also the vector fields $\mathcal{W}^1(\Gamma, \Lambda)(X)$ and $\mathcal{J}^1(\Gamma[F], \Lambda)(X)$ are φ_A-related for all $A \in T^1_mF$ and all $X \in C^\infty TM$. This implies our claim. Indeed, the lifting map of $\mathcal{W}^1(\Gamma, \Lambda)$ can be constructed as follows. For every $\xi \in T_xM$ we take a vector field X on M such that $j^1_xX = \Lambda(\xi)$. Then the lifts of ξ with respect to $\mathcal{W}^1(\Gamma, \Lambda)$ coincide with the values of $\mathcal{W}^1(\Gamma X)$ along W^1_xP. The same is true for the lifting map of $J^1(\Gamma[F], \Lambda)$. Hence the vector fields $\mathcal{W}^1(\Gamma, \Lambda)(X)$ and $\mathcal{J}^1(\Gamma[F], \Lambda)(X)$ are φ_A-related. □

There is another construction transforming a general connection Γ in $Y \to M$ and a linear connection $\Lambda : TM \to J^1TM$ into a connection $P(\Gamma, \Lambda)$ in $J^1Y \to M$, see 45.7 of [12]. Since $J^1Y \to Y$ is an affine bundle with the modelling vector bundle $VY \otimes T^*M$, where VY is the vertical tangent bundle of Y, $\Gamma : Y \to J^1Y$ defines an identification $I_\Gamma : J^1Y \approx VY \otimes T^*M$. The connection Γ in Y gives rise to a connection

$\mathcal{V}\Gamma$ in $VY \to M$, which is linear over Y (see [12], p. 225). Hence one can construct the tensor product $\mathcal{V}\Gamma \otimes \Lambda^*$ with the dual connection Λ^* on T^*M. The identification I_Γ transforms $\mathcal{V}\Gamma \otimes \Lambda^*$ into a connection $P(\Gamma, \Lambda)$ in $J^1Y \to M$.

In particular, let Γ be a principal connection in $P(M, G)$, Λ a principal connection in P^1M and F a left G-space. On one hand, we have the connection $p(\Gamma, \Lambda)$ in W^1P and the induced connection $p(\Gamma, \Lambda)[T^1_m F]$ in J^1E, $E = P[F]$. On the other hand, we can construct $P(\Gamma[F], \Lambda)$. The following claim is analogous to Proposition 7.

Proposition 8. $p(\Gamma, \Lambda)[T^1_m F] = P(\Gamma[F], \Lambda)$.

Proof. Let $l : G \times F \to F$ be the left action of G on F. By Theorem 10.18 of [12], VE is a fiber bundle associated to P with standard fiber TF with respect to the action $T_2 l : G \times TF \to TF$, where T_2 is the second partial tangent functor. Since $R(\Gamma)$ is a reduction of W^1P, we may consider J^1E as a fiber bundle associated to $R(\Gamma) = P^1M \times_M P$. Following the definition of $p(\Gamma, \Lambda)$ we have to prove that $P(\Gamma, \Lambda)$ is the connection induced from $\Lambda \times \Gamma$. We have $T^1_m F = TF \otimes \mathbb{R}^{m*}$. The restricted action of $G^1_m \times i(G)$ on $T^1_m F$ identifies the associated bundle $R(\Gamma)[T^1_m F]$ with $VE \otimes T^*M$. One verifies easily that this is the identification of J^1E with $VE \otimes T^*M$ which is determined by $\Gamma[F]$. The result follows then from the simple fact that a connection $\Lambda \times \Gamma$ induces the connection $\mathcal{V}(\Gamma[F]) \otimes \Lambda^*$ in $VE \otimes T^*M$. □

We are now in position to prove Proposition 6. From the formulae on p. 366 in [12] it follows that $P(\Gamma[F], \Lambda) - \mathcal{J}^1(\Gamma[F], \widetilde{\Lambda})$ is the pullback of the curvature $C(\Gamma)$ to the pullback of VY over J^1Y. Consider the special case $F = G$. Then the difference $p(\Gamma, \Lambda) - \mathcal{W}^1(\Gamma, \widetilde{\Lambda})$ is projected onto $P(\Gamma[G], \Lambda) - \mathcal{J}^1(\Gamma[G], \widetilde{\Lambda})$. This implies $p(\Gamma, \Lambda) - \mathcal{W}^1(\Gamma, \widetilde{\Lambda}) = C(\Gamma)$.

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I. Kolář, Department of Algebra and Geometry, Masaryk University, Janáčkovo nám. 2a, 66295 Brno, Czech Republic
G. Virsik, Department of Mathematics, Monash University, Clayton, Victoria 3168, Australia