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ON GEODESIC MAPPINGS OF SPECIAL FINSLER SPACES

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ABSTRACT. In an earlier paper [2] there arose an interesting problem: Determine all the Finsler spaces which have common geodesics with some Riemannian space, that is, determine all the Finsler spaces which admit a geodesic mapping onto a Riemannian space. Such Finsler spaces have vanishing Douglas tensor, and are called Douglas spaces [3]. In the present paper we shall give some special examples of geodesic mappings between a Finsler space and a Riemannian space.

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1. INTRODUCTION

Let $F^n(M^n, L)$ be an n-dimensional Finsler space, where M^n is a connected differentiable manifold of dimension n and $L(x, y)$, where $y^i = \dot{x}^{i1}$, is the fundamental function defined on the manifold $TM\backslash O$ of nonzero tangent vectors. (Throughout the present lecture we shall use the terminology and definitions described in Matsumoto's monograph [8].)

The system of differential equations for geodesic curves of F^n with respect to the canonical parameter t is given by $\ddot{x}^i + 2G^i(x, y) = 0$, where

$$G^i = \tfrac{1}{4} g^{ir}(y^m \partial L^2_{(r)}/\partial x^m - \partial L^2/\partial x^r),$$

and $g^{ij} = (g_{ij})^{-1}$, $g_{ij} = \frac{1}{2} L^2_{(i)(j)}$, $L_{(i)} = \partial L/\partial y^i$. The Berwald connection coefficients $G^i(x, y), G^i_{jk}(x, y)$ can be derived from the functions G^i, namely $G^i_j = G^i_{(j)}$; $G^i_{jk} = G^i_{j(k)}$.

Let us consider two Finsler spaces $F^n(M^n, L)$ and $\bar{F}^n(M^n, \bar{L})$ and a common underlying manifold. A diffeomorphism $F^n \to \bar{F}^n$ is called *geodesic* if it maps an arbitrary geodesic of F^n to a geodesic of $\bar{F}^n$. In this case the change $L - \bar{L}$ of the metrics is called *projective*. As it is well known, the mapping $F^n \to \bar{F}^n$ is geodesic if and only if there exists a scalar field $p(x, y)$ satisfying

$$\bar{G}^i = G^i + py^i; \qquad p \neq 0. \tag{1.1}$$

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[1] The Roman indices run over the range $1, \dots, n$.

The projective factor $p(x, y)$ is a positively homogeneous function of degree one in y. From (1.1) we have

$$\bar{G}^i_j = G^i_j + p\delta^i_j + p_j y^i, \tag{1.2}$$

$$\bar{G}^i_{jk} = G^i_{jk} + p_j\delta^i_k + p_k\delta^i_j + p_{jk}y^i, \tag{1.3}$$

where $p_j = p_{(j)}$ and $p_{jk} = p_{j(k)}$.

Using the Rapcsák paper [10] M. Matsumoto obtained the following result [9]:

"If a Finsler space $F^n = (M^n, L)$ is projective to a Finsler space $\bar{F}_n = (M^n, \bar{L})$ then

$$\bar{l}_{ij;r}y^r = 0, \tag{1.4}$$

where $\bar{l}_{ij} = \frac{1}{L}\bar{h}_{ij} = \frac{1}{L}(\bar{g}_{ij} - \bar{l}_i\bar{l}_j)$ and $\bar{l}_i = \bar{L}_{(i)}$."

The symbol "; " denotes the h-covariant derivative with respect to the Berwald connection $B\Gamma = (G^i_{jk}, G^i_j)$ in F^n. The purpose of the present paper is to study equation (1.4) in some special cases, and to investigate the geodesic maps between Finsler and Riemannian spaces.

2. On the equation $\bar{l}_{ij;r}y^r = 0$

Differentiating (1.4) by y^k we have

$$\bar{l}_{ij;r(k)}y^r + \bar{l}_{ij;k} = 0. \tag{2.1}$$

Using the Ricci identities

$$\bar{l}_{ij;r(k)} - \bar{l}_{ij(k);r} = -\bar{l}_{mj}G^m_{irk} - \bar{l}_{im}G^m_{jrk},$$

after transvecting by y^r we obtain

$$\bar{l}_{ij;r(k)}y^r - \bar{l}_{ij(k);r}y^r = 0. \tag{2.2}$$

From (2.1) and (2.2) follows that

$$\bar{l}_{ij(k);r}y^r + \bar{l}_{ij;k} = 0.$$

This equation may be written in the form

$$\left[-\tfrac{1}{L^2}\bar{l}_k\bar{h}_{ij} + \tfrac{2}{L}\bar{C}_{ijk} - \tfrac{1}{L^2}(\bar{h}_{ik}\bar{l}_j + \bar{h}_{jk}\bar{l}_i)\right]_{;r} y^r = -\bar{l}_{ij;k}. \tag{2.3}$$

Applying (1.2) and (1.3) we get

$$\left(\tfrac{2}{L}\bar{C}_{ijk}\right)_{;r} y^r = -\tfrac{2}{L}p\bar{C}_{ijk} + \tfrac{2}{L}\bar{P}_{ijk},$$

where $\bar{C}_{ijk} = \frac{1}{2}g_{ij(k)}$ and $\bar{C}_{ijk;r}y^r = \bar{P}_{ijk}$. Thus (2.3) may be written in the form

$$\bar{l}_{ij}\bar{N}_k + \bar{l}_{ik}\bar{N}_j + \bar{l}_{jk}\bar{N}_i + \tfrac{2}{L}p\bar{C}_{ijk} - \tfrac{2}{L}\bar{P}_{ijk} = \bar{l}_{ij;k} \tag{2.4}$$

where $\bar{N}_i = \bar{M}_{i;r}y^r$ and $\bar{M}_i = \frac{1}{L}\bar{l}_i$, which gives

Proposition 1. *In the case of a geodesic mapping of Finsler spaces F^n and $\bar{F}^n$ the tensor $\bar{l}_{ij;k}$ is symmetric in all indices.*

Example 1. We consider the Randers change $\bar{L}(x,y) = L(x,y) + \beta(x,y)$, where $\beta(x,y)$ is a closed one-form, then this change $L \to \bar{L}$ is projective. Thus we get $\frac{1}{L}h_{ij} = \frac{1}{\bar{L}}\bar{h}_{ij}$, that is $\bar{l}_{ij} = l_{ij}$.

Differentiating this equation covariantly with respect to $B\Gamma$ in F_n we obtain

$$\bar{l}_{ij;k} = l_{ij;k} = -\tfrac{2}{L}P_{ijk}.$$

Thus in the case of Randers change the equation (2.4) can be rewritten in the form

$$\bar{l}_{ij}\bar{N}_k + \bar{l}_{ik}\bar{N}_j + \bar{l}_{jk}\bar{N}_i + \tfrac{2}{\bar{L}}p\bar{C}_{ijk} - \tfrac{2}{\bar{L}}\bar{P}_{ijk} = -\tfrac{2}{L}P_{ijk}.$$

We assume that F^n is a Landsberg space ($P_{ijk} = 0$) then we get

$$\bar{N}_i = \tfrac{2}{(n+1)\bar{L}}(\bar{P}_i - p\bar{C}_i),$$

where $\bar{P}_i = \bar{P}_{ijk}\bar{g}^{jk}$; $\bar{C}_i = \bar{C}_{ijk}\bar{g}^{jk}$.

At first M. Matsumoto [6], [7] studied the special Finsler space satisfying the condition $P_{ijk} = \lambda(x,y)C_{ijk}$, and after him M. Hashiguchi [4] and H. Izumi [5]. It is well-known that this condition is satisfied in all two-dimensional Finsler spaces. If we consider the Finsler space $\bar{F}_n$ fulfilling the condition $\bar{P}_{ijk} = p\bar{C}_{ijk}$, then we get

$$\bar{N}_i = \bar{M}_{i;r}y^r = 0 \qquad \text{that is}$$

$$\left(\tfrac{1}{\bar{L}}\bar{l}_i\right)_{;r} y^r = \left[\frac{1}{\bar{L}}\bar{l}_{i;r} - \frac{1}{\bar{L}^2}\bar{l}_i\bar{L}_{;r}\right] y^r = 0. \tag{2.5}$$

Using the equations (1.1), (1.2) and (1.3) we obtain

$$\begin{aligned}\bar{L}_{;r}y^r &= 2p\bar{L},\\ \bar{l}_{i;r}y^r &= \bar{l}_ip + \bar{L}p_i.\end{aligned}$$

So, we have

$$\tfrac{1}{\bar{L}}(\bar{l}_ip + \bar{L}p_i) - \tfrac{1}{\bar{L}^2}\bar{l}_i2p\bar{L} = 0,$$

from which it follows that

$$\frac{p_i}{p} - \frac{\bar{l}_i}{\bar{L}} = 0,$$

which yields

$$p(x,y) = e^{\varphi(x)}\bar{L}(x,y). \tag{2.6}$$

Thus we have proved

Proposition 2. *If we suppose that there exists a geodesic map (Randers change with respect to projective scalar $p(x,y)$) between a Landsberg and a Finsler space fulfilling the condition $\bar{P}_{ijk} = p(x,y)\bar{C}_{ijk}$, then $p(x,y)$ is given by the equation* (2.6).

Question. In which Finsler spaces $\bar{F}^n$ does the condition $\bar{P}_{ijk} = \lambda(x,y)\bar{C}_{ijk}$ hold where $\lambda(x,y) = \sigma(x)\bar{L}$ and $\sigma(x)$ depend on the position only?

Remark [5]. For $n > 3$, in a C-reducible Finsler space with the condition $\bar{P}_{ijk} = \lambda(x,y)\bar{C}_{ijk}$ we have $\lambda(x,y) = \sigma(x)\bar{L}$, where $\sigma(x)$ depends on the position only.

Example 2. From (1.1), (1.2) and (1.3) we can easily obtain the following well-known relation between the $(v)h$-torsion tensors of F^n on $\bar{F}^n$:

$$\bar{R}^h_{ij} = R^h_{ij} + y^h Q_{ij} + \delta^h_i Q_j + \delta^h_j Q_i,$$

where $Q_i = p_{;i} - pp_i$ and $Q_{ij} = p_{i;j} - p_{j;i}$.

Now we assume that $\bar{P}_{ijk} = p(x,y)\bar{C}_{ijk}$ in $\bar{F}^n$, and F^n is a Finsler space of constant curvature. Using the integrability condition of the equation (2.4) we get the following equation

$$K\left(Ll_k\frac{\bar{l}_l}{\bar{L}} - Ll_l\frac{\bar{l}_k}{\bar{L}}\right) = \frac{\bar{l}_l}{\bar{L}}Q_k - \frac{\bar{l}_k}{\bar{L}}Q_l = (\bar{N}_{k;l} - \bar{N}_{l;k}) \tag{2.7}$$

from which we get

$$(KLl_k - Q_k)\frac{\bar{l}_l}{\bar{L}} = (KLl_l - Q_l)\frac{\bar{l}_k}{\bar{L}} \tag{2.8}$$

where K is the curvature constant in F_n.

3. On the strongly geodesic mapping

Definition. If a geodesic mapping satisfies the condition $\bar{l}_{ij;k} = 0$, the mapping is called a *strongly geodesic mapping*.

Now we consider a geodesic mapping between a Finsler (F^n) and a Riemannian (R^n) space. Then from (2.4) we get

$$\bar{l}_{ij}\bar{N}_k + \bar{l}_{ik}\bar{N}_j + \bar{l}_{jk}\bar{N}_i = \bar{l}_{ij;k} \tag{3.1}$$

This equation is satisfied in the case of a geodesic mapping of a Berwald space on a Riemannian space. We can easily show

Proposition 3. *$\bar{l}_{ij;k} = 0$ holds good if and only if $\bar{N}_i = 0$.*

From (2.5) we obtain

Proposition 4. *In the case of a strongly geodesic mapping $F^n \to R^n$ the projective scalar function $p(x,y) = e^{\varphi(x)}\bar{L}(x,y)$.*

The equation (2.7) yields

$$K\left(Ll_k\frac{\bar{l}_l}{\bar{L}} - Ll_l\frac{\bar{l}_k}{\bar{L}}\right) = 0.$$

Contracting this by y^l we obtain

$$K\left(\frac{l_k}{L} - \frac{\bar{l}_k}{\bar{L}}\right) = 0.$$

For a Finsler space of constant curvature we have the following

Theorem. *If a change $F^n(M^n, L) \to R^n(M^n, \bar{L})$ is strongly projective and F^n is a Finsler space of constant curvature, then we have two cases*

(a) $K = 0$
(b) $K \neq 0$ *and* $\bar{L} = e^{\varphi(x)}L$.

From Rund's [11] and Aikou's [1] result follows that in the case (b) we get a homothetic mapping.

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