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ON GEODESIC MAPPINGS OF SPECIAL FINSLER SPACES

SÁNDOR BÁCSÓ

ABSTRACT. In an earlier paper [2] there arose an interesting problem: Determine all the Finsler spaces which have common geodesics with some Riemannian space, that is, determine all the Finsler spaces which admit a geodesic mapping onto a Riemannian space. Such Finsler spaces have vanishing Douglas tensor, and are called Douglas spaces [3]. In the present paper we shall give some special examples of geodesic mappings between a Finsler space and a Riemannian space.

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1. INTRODUCTION

Let $F^n(M^n, L)$ be an *n*-dimensional Finsler space, where M^n is a connected differentiable manifold of dimension *n* and L(x, y), where $y^i = \dot{x}^{i1}$, is the fundamental function defined on the manifold $TM \setminus O$ of nonzero tangent vectors. (Throughout the present lecture we shall use the terminology and definitions described in Matsumoto's monograph [8].)

The system of differential equations for geodesic curves of F^n with respect to the canonical parameter t is given by $\ddot{x}^i + 2G^i(x, y) = 0$, where

$$G^{i} = \frac{1}{4}g^{ir}(y^{m}\partial L^{2}_{(r)}/\partial x^{m} - \partial L^{2}/\partial x^{r}),$$

and $g^{ij} = (g_{ij})^{-1}$, $g_{ij} = \frac{1}{2}L^2_{(i)(j)}$, $L_{(i)} = \partial L/\partial y^i$. The Berwald connection coefficients $G^i(x,y), G^i_{jk}(x,y)$ can be derived from the functions G^i , namely $G^i_j = G^i_{(j)}; G^i_{jk} = G^i_{(jk)}$.

Let us consider two Finsler spaces $F^n(M^n, L)$ and $\overline{F}^n(M^n, \overline{L})$ and a common underlying manifold. A diffeomorphism $F^n \to \overline{F}^n$ is called *geodesic* if it maps an arbitrary geodesic of F^n to a geodesic of \overline{F}^n . In this case the change $L - \overline{L}$ of the metrics is called *projective*. As it is well known, the mapping $F^n \to \overline{F}^n$ is geodesic if and only if there exists a scalar field p(x, y) satisfying

(1.1)
$$\bar{G}^i = G^i + py^i; \qquad p \neq 0.$$

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¹The Roman indices run over the range $1, \ldots, n$.

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The projective factor p(x, y) is a positively homogeneous function of degree one in y. From (1.1) we have

(1.2)
$$\bar{G}^i_j = G^i_j + p\delta^i_j + p_j y^i,$$

(1.3)
$$\bar{G}^i_{jk} = G^i_{jk} + p_j \delta^i_k + p_k \delta^i_j + p_{jk} y^i,$$

where $p_j = p_{(j)}$ and $p_{jk} = p_{j(k)}$.

Using the Rapcsák paper [10] M. Matsumoto obtained the following result [9]:

"If a Finsler space $F^n = (M^n, L)$ is projective to a Finsler space $\overline{F}_n = (M^n, \overline{L})$ then

where $\bar{l}_{ij} = \frac{1}{\bar{L}} \bar{h}_{ij} = \frac{1}{\bar{L}} (\bar{g}_{ij} - \bar{l}_i \bar{l}_j)$ and $\bar{l}_i = \bar{L}_{(i)}$." The symbol "; " denotes the *h*-covariant derivative with respect to the Berwald connection $B\Gamma = (G_{ik}^i, G_i^i)$ in F^n . The purpose of the present paper is to study equation (1.4) in some special cases, and to investigate the geodesic maps between Finsler and Riemannian spaces.

2. On the equation $\bar{l}_{ij;r}y^r = 0$

Differentiating (1.4) by y^k we have

 $\bar{l}_{ij;r(k)}y^r + \bar{l}_{ij;k} = 0.$ (2.1)

Using the Ricci identities

$$\bar{l}_{ij;r(k)} - \bar{l}_{ij(k);r} = -\bar{l}_{mj}G^m_{irk} - \bar{l}_{im}G^m_{jrk},$$

after transvecting by y^r we obtain

 $\bar{l}_{ij;r(k)}y^r - \bar{l}_{ij(k);r}y^r = 0.$ (2.2)

From (2.1) and (2.2) follows that

$$\bar{l}_{ij(k);r}y^r + \bar{l}_{ij;k} = 0.$$

This equation may be written in the form

(2.3)
$$\left[-\frac{1}{L^2}\bar{l}_k\bar{h}_{ij} + \frac{2}{L}\bar{C}_{ijk} - \frac{1}{L^2}(\bar{h}_{ik}\bar{l}_j + \bar{h}_{jk}\bar{l}_i)\right]_{;r}y^r = -\bar{l}_{ij;k}$$

Applying (1.2) and (1.3) we get

$$\left(\tfrac{2}{L}\bar{C}_{ijk}\right)_{;r}y^{r}=-\tfrac{2}{L}p\bar{C}_{ijk}+\tfrac{2}{L}\bar{P}_{ijk},$$

where $\bar{C}_{ijk} = \frac{1}{2}g_{ij(k)}$ and $\bar{C}_{ijk;r}y^r = \bar{P}_{ijk}$. Thus (2.3) may be written in the form

(2.4)
$$\bar{l}_{ij}\bar{N}_k + \bar{l}_{ik}\bar{N}_j + \bar{l}_{jk}\bar{N}_i + \frac{2}{L}p\bar{C}_{ijk} - \frac{2}{L}\bar{P}_{ijk} = \bar{l}_{ij;k}$$

where $\bar{N}_i = \bar{M}_{i;r} y^r$ and $\bar{M}_i = \frac{1}{\bar{I}} \bar{l}_i$, which gives

Proposition 1. In the case of a geodesic mapping of Finsler spaces F^n and \overline{F}^n the tensor $l_{ij:k}$ is symmetric in all indices.

Example 1. We consider the Randers change $\bar{L}(x,y) = L(x,y) + \beta(x,y)$, where $\beta(x,y)$ is a closed one-form, then this change $L \to \bar{L}$ is projective. Thus we get $\frac{1}{L}h_{ij} = \frac{1}{L}\bar{h}_{ij}$, that is $\bar{l}_{ij} = l_{ij}$.

Differentiating this equation covariantly with respect to $B\Gamma$ in F_n we obtain

$$\bar{l}_{ij;k} = l_{ij;k} = -\frac{2}{L}P_{ijk}$$

Thus in the case of Randers change the equation (2.4) can be rewritten in the form

$$\bar{l}_{ij}\bar{N}_k + \bar{l}_{ik}\bar{N}_j + \bar{l}_{jk}\bar{N}_i + \frac{2}{L}p\bar{C}_{ijk} - \frac{2}{L}\bar{P}_{ijk} = -\frac{2}{L}P_{ijk}.$$

We assume that F^n is a Landsberg space $(P_{ijk} = 0)$ then we get

$$\bar{N}_i = \frac{2}{(n+1)\bar{L}} (\bar{P}_i - p\bar{C}_i),$$

where $\bar{P}_i = \bar{P}_{ijk}\bar{g}^{jk}$; $\bar{C}_i = \bar{C}_{ijk}\bar{g}^{jk}$.

At first M. Matsumoto [6], [7] studied the special Finsler space satisfying the condition $P_{ijk} = \lambda(x, y)C_{ijk}$, and after him M. Hashiguchi [4] and H. Izumi [5]. It is well-known that this condition is satisfied in all two-dimensional Finsler spaces. If we consider the Finsler space \bar{F}_n fulfilling the condition $\bar{P}_{ijk} = p\bar{C}_{ijk}$, then we get

(2.5)
$$\bar{N}_{i} = \bar{M}_{i;r}y^{r} = 0 \quad \text{that is}$$
$$\left(\frac{1}{\bar{L}}\bar{l}_{i}\right)_{;r}y^{r} = \left[\frac{1}{\bar{L}}\bar{l}_{i;r} - \frac{1}{\bar{L}^{2}}\bar{l}_{i}\bar{L}_{;r}\right]y^{r} = 0.$$

Using the equations (1.1), (1.2) and (1.3) we obtain

$$ar{L}_{;r}y^r=2par{L},\ ar{l}_{i;r}y^r=ar{l}_ip+ar{L}p_i.$$

So, we have

$$\frac{1}{\bar{L}}(\bar{l}_ip+\bar{L}p_i)-\frac{1}{\bar{L}^2}\bar{l}_i2p\bar{L}=0,$$

from which it follows that

$$\frac{p_i}{p} - \frac{\bar{l}_i}{\bar{L}} = 0,$$

which yields

$$(2.6) p(x,y) = e^{\varphi(x)} \overline{L}(x,y)$$

Thus we have proved

Proposition 2. If we suppose that there exists a geodesic map (Randers change with respect to projective scalar p(x, y)) between a Landsberg and a Finsler space fulfilling the condition $\bar{P}_{ijk} = p(x, y)\bar{C}_{ijk}$, then p(x, y) is given by the equation (2.6).

Question. In which Finsler spaces \bar{F}^n does the condition $\bar{P}_{ijk} = \lambda(x, y)\bar{C}_{ijk}$ hold where $\lambda(x, y) = \sigma(x)\bar{L}$ and $\sigma(x)$ depend on the position only?

Remark [5]. For n > 3, in a *C*-reducible Finsler space with the condition $\bar{P}_{ijk} =$ $\lambda(x,y)\overline{C}_{ijk}$ we have $\lambda(x,y) = \sigma(x)\overline{L}$, where $\sigma(x)$ depends on the position only.

Example 2. From (1.1), (1.2) and (1.3) we can easily obtain the following well-known relation between the (v)h-torsion tensors of F^n on \overline{F}^n :

$$\bar{R}^h_{ij} = R^h_{ij} + y^h Q_{ij} + \delta^h_i Q_j + \delta^h_j Q_i,$$

where $Q_i = p_{;i} - pp_i$ and $Q_{ij} = p_{i;j} - p_{j;i}$. Now we assume that $\bar{P}_{ijk} = p(x, y)\bar{C}_{ijk}$ in \bar{F}^n , and F^n is a Finsler space of constant curvature. Using the integrability condition of the equation (2.4) we get the following equation

(2.7)
$$K\left(Ll_{k}\frac{\bar{l}_{l}}{L}-Ll_{l}\frac{\bar{l}_{k}}{L}\right)=\frac{\bar{l}_{l}}{L}Q_{k}-\frac{\bar{l}_{k}}{L}Q_{l}=(\bar{N}_{k;l}-\bar{N}_{l;k})$$

from which we get

(2.8)
$$(KLl_k - Q_k) \frac{\overline{l}_l}{L} = (KLl_l - Q_l) \frac{\overline{l}_k}{L}$$

where K is the curvature constant in F_n .

3. On the strongly geodesic mapping

Definition. If a geodesic mapping satisfies the condition $\bar{l}_{ij,k} = 0$, the mapping is called a strongly geodesic mapping.

Now we consider a geodesic mapping between a Finsler (F^n) and a Riemannian (\mathbb{R}^n) space. Then from (2.4) we get

$$(3.1) \qquad \qquad \bar{l}_{ij}\bar{N}_k + \bar{l}_{ik}\bar{N}_j + \bar{l}_{jk}\bar{N}_i = \bar{l}_{ij;k}$$

This equation is satisfied in the case of a geodesic mapping of a Berwald space on a Riemannian space. We can easily show

Proposition 3. $\bar{l}_{ij;k} = 0$ holds good if and only if $\bar{N}_i = 0$.

From (2.5) we obtain

Proposition 4. In the case of a strongly geodesic mapping $F^n \to R^n$ the projective scalar function $p(x, y) = e^{\varphi(x)} \overline{L}(x, y)$.

The equation (2.7) yields

$$K\left(Ll_{k}\frac{\bar{l}_{l}}{L}-Ll_{l}\frac{\bar{l}_{k}}{L}\right)=0.$$

Contracting this by y^{l} we obtain

$$K\left(\frac{l_k}{L}-\frac{\bar{l}_k}{L}\right)=0.$$

For a Finsler space of constant curvature we have the following

Theorem. If a change $F^n(M^n, L) \to R^n(M^n, \overline{L})$ is strongly projective and F^n is a Finsler space of constant curvature, then we have two cases

- (a) K = 0
- (b) $K \neq 0$ and $\overline{L} = e^{\varphi(x)}L$.

From Rund's [11] and Aikou's [1] result follows that in the case (b) we get a homothetic mapping.

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