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NATURAL LIFTING OF CONNECTIONS TO VERTICAL BUNDLES

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ABSTRACT. First we study the flow prolongation of projectable vector fields with respect to a bundle functor of order (r, s, q) on the category of fibered manifolds. Using this approach, we construct an operator transforming connections on a fibered manifold Y into connections on an arbitrary vertical bundle over Y. Then we deduce that this operator is the only natural one of finite order and we present a condition on vertical bundles over Y under which every natural operator in question has finite order.

An important result in the theory of general connections on an arbitrary fibered manifold $Y \to M$ is that every connection Γ on Y induces naturally a unique connection $\mathcal{V}\Gamma$ on the vertical tangent bundle $VY \to M$, [3]. The starting point for the present paper is the fact that the vertical tangent functor V is the vertical modification of the tangent functor T. We replace T by an arbitrary bundle functor F on the category $\mathcal{M}f_n$ of n-dimensional manifolds and their local diffeomorphisms, [3], and we consider its vertical modification V^F. Our main result is Proposition 2, which reads that there is a unique natural operator of finite order transforming connection on $Y \to M$ into connections on $V^F Y \to M$. In Proposition 3, we present a condition on F under which every natural operator in question has finite order.

To construct one operator of this type, we use the flow prolongation of projectable vector fields on Y with respect to a bundle functor on the category $\mathcal{FM}$ of fibered manifolds and their morphisms. Recently, it has been clarified that the jets of $\mathcal{FM}$-morphisms are characterized by a triple of integers (r, s, q), $s \geq r$, $q \geq r$, [1]. So even the order of bundle functors on $\mathcal{FM}$ is to be characterized by such triples. In Proposition 1 we deduce that the flow prolongation of a projectable vector field Z on Y with respect to a bundle functor of order (r, s, q) depends on the (r, s, q)-jets of Z. Then we combine lifting of vector fields on M with respect to a connection Γ on Y

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with the flow prolongation. For the special case $r = q$, this construction was studied in [3], p.364. In the case of V^F, we obtain in this way one natural operator $\mathcal{V}^F$ of finite order transforming connections from $Y \to M$ to $V^F Y \to M$. So the proof of Proposition 2 deals with the uniqueness problem only. In the last section we point out that in the case F is a Weil functor T^A there is another natural construction of an induced connection on $V^A Y \to M$, which is based on a canonical exchange map from [2]. By the uniqueness in Proposition 2, the result of the second construction must coincide with the first one.

All manifolds and maps are assumed to be infinitely differentiable. Unless otherwise specified, we use the terminology and notation from [3].

1. Bundle functors of order (r, s, q).

Let $p : Y \to M$ and $\overline{p} : \overline{Y} \to \overline{M}$ be two fibered manifolds and r, $s \geq r$, $q \geq r$ be integers. We recall that two $\mathcal{FM}$-morphisms $f, g : Y \to \overline{Y}$ with the base maps $\underline{f}, \underline{g} : M \to \overline{M}$ determine the same (r, s, q)-jet $j_y^{r,s,q} f = j_y^{r,s,q} g$ at $y \in Y$, $p(y) = x$, if

$$j_y^r f = j_y^r g, \; j_y^s(f|Y_x) = j_y^s(g|Y_x), \; j_x^q \underline{f} = j_x^q \underline{g} \; .$$

The space of all such (r, s, q)-jets is denoted by $J^{r,s,q}(Y, \overline{Y})$. The composition of $\mathcal{FM}$-morphisms induces the composition of (r, s, q)-jets, [3], p.116.

Write $\mathbf{R}^{m,n} = (pr_1 : \mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}^m)$ for the product fibered manifold. If $m = dim(M)$ and $m + n = dim(Y)$, we introduce the principal bundle of all (r, s, q)-frames on Y by

$$P^{r,s,q} Y = inv J_{0,0}^{r,s,q}(\mathbf{R}^{m,n}, Y) \; ,$$

where inv indicates the invertible (r, s, q)-jets and $(0, 0) \in \mathbf{R}^m \times \mathbf{R}^n$. Its structure group is

$$G_{m,n}^{r,s,q} = inv J_{0,0}^{r,s,q}(\mathbf{R}^{m,n}, \mathbf{R}^{m,n})_{0,0}$$

and both multiplication in $G_{m,n}^{r,s,q}$ and the right action of $G_{m,n}^{r,s,q}$ on $P^{r,s,q} Y$ is given by the jet composition.

Let $\mathcal{FM}_{m,n}$ be the category of fibered manifolds with m-dimensional bases and n-dimensional fibers and their local isomorphisms.

Definition 1. A bundle functor F on $\mathcal{FM}_{m,n}$ is said to be of order (r, s, q), if $j_y^{r,s,q} f = j_y^{r,s,q} g$ implies $Ff|F_y Y = Fg|F_y Y$.

This definition implies that the standard fiber $S = F_{0,0}(\mathbf{R}^{m,n})$ of F is a left $G_{m,n}^{r,s,q}$-space. Quite similarly to the classical case, [3], one deduces that the bundle functors of order (r, s, q) on $\mathcal{FM}_{m,n}$ are in bijection with the left actions of $G_{m,n}^{r,s,q}$.

A projectable vector field $Z : Y \to TY$ is an $\mathcal{FM}$-morphism over the underlying vector field $M \to TM$. Its flow $exptZ$ is formed by local $\mathcal{FM}_{m,n}$-morphisms. If F is a bundle functor on $\mathcal{FM}_{m,n}$, the flow prolongation of Z with respect to F is defined by

$$\mathcal{F}Z = \frac{\partial}{\partial t}|_0 F(exptZ) \; . \tag{1}$$

This map is $\mathbf{R}$-linear and preserves bracket, [3].

Proposition 1. *If F is of order (r,s,q), then the value of $\mathcal{F}Z$ at each point of F_yY depends on $j_y^{r,s,q}Z$ only.*

Proof. Let $\varphi : G_{m,n}^{r,s,q} \times S \to S$ be the action defining F. The translations on $\mathbf{R}^m \times \mathbf{R}^n$ define two identifications $F\mathbf{R}^{m,n} = \mathbf{R}^{m,n} \times S$ and

$$invJ^{r,s,q}(\mathbf{R}^{m,n}, \mathbf{R}^{m,n}) = \mathbf{R}^{m,n} \times G_{m,n}^{r,s,q} \times \mathbf{R}^{m,n} \ . \tag{2}$$

The effect of F on an $\mathcal{FM}_{m,n}$-morphism $f : \mathbf{R}^{m,n} \to \mathbf{R}^{m,n}$ is given by

$$Ff(y,a) = (f(y), \varphi((j_y^{r,s,q}f)_2, a)) \ , y \in \mathbf{R}^{m,n} \ , \ a \in S \ , \tag{3}$$

where $(j_y^{r,s,q}f)_2$ means the second component in the decomposition (2). If we insert the flow of Z into (3) and evaluate $\frac{\partial}{\partial t}|_0$, we prove our claim. □

Thus the construction of the flow prolongation can be interpreted as a map

$$\mathcal{F}_Y : FY \times_Y J^{r,s,q}TY \to TFY \ ,$$

where $J^{r,s,q}TY$ denotes the space of all (r,s,q)-jets of projectable vector fields on Y. Since the flow prolongation is $\mathbf{R}$-linear, $\mathcal{F}_Y$ is linear in the second factor.

2. Lifting of connections

Let Γ be a connection on Y with the coordinate expression

$$dy^p = \Gamma_i^p(x,y)dx^i \ . \tag{4}$$

If X is a vector field on M with the coordinate components $X^i(x)$, then its lift ΓX is a vector field on Y, whose coordinate form is

$$X^i(x)\frac{\partial}{\partial x^i} + \Gamma_i^p(x,y)X^i(x)\frac{\partial}{\partial y^p} \ . \tag{5}$$

By Proposition 1, $\mathcal{F}(\Gamma X)$ depends on the q-jets of X only. So we obtain a map

$$\mathcal{F}\Gamma : FY \times_M J^qTM \to TFY \ , \tag{6}$$

which is linear in the second factor. If $q = 0 = r$, then (6) is a connection on FY.

In general, let $\Lambda : TM \to J^qTM$ be a linear q-th order connection on M, i.e. a linear splitting of the jet projection $J^qTM \to TM$. By linearity, the composition

$$\mathcal{F}(\Gamma, \Lambda) = \mathcal{F}\Gamma \circ (id_{FY} \times_{id_M} \Lambda) : FY \times_M TM \to TFM$$

is the lifting map of a connection on $FY \to M$.

In what follows we start from a bundle functor on $\mathcal{M}f_n$ of order s, which will be also denoted by F. Its vertical modification V^F is a bundle functor on $\mathcal{FM}_{m,n}$ defined by

$$V^FY = \bigcup_{x \in M} F(Y_x) \ , \ V^Ff = \bigcup_{x \in M} F(f_x) \ ,$$

where f_x is the restriction and corestriction of $f : Y \to \overline{Y}$ over $\underline{f} : M \to \overline{M}$ to the fibers Y_x and $\overline{Y}_{\underline{f}(x)}$, [4]. For $F = T$, we use the standard notation V instead of V^T. Clearly, the order of V^F is $(0,s,0)$. By (6), we have defined $\mathcal{V}^F\Gamma$ for every connection Γ on Y.

Definition 2. The connection $\mathcal{V}^F\Gamma$ is called the F-vertical prolongation of Γ.

3. Natural operators

The following naturality property of $\mathcal{V}^F\Gamma$ is an interesting generalization of the well known result concerning $\mathcal{V}\Gamma$, [3], p. 255, to an arbitrary bundle functor F on $\mathcal{M}f_n$.

Proposition 2. *$\mathcal{V}^F$ is the only natural operator of finite order transforming connections on $Y \to M$ into connections on $V^FY \to M$.*

Proof. The technical part of the proof is heavily based on the methods for finding natural operators developed in [3]. So we shall use freely the standard notation from this book.

If D_1 and D_2 are two natural operators of our type, then the difference $\Delta\Gamma = D_1\Gamma - D_2\Gamma$ is a natural tensor field $V^FY \to V(V^FY) \otimes T^*M$. We are going to deduce Δ is the zero tensor field. Write r for the maximum of s and of the order of Δ. Let $S^r = (y^p_{i\alpha\beta})$, $|\alpha| + |\beta| \le r$, be the space of r-jets of connections on $\mathbf{R}^{m,n}$ over $(0,0)$, where the multiindex α corresponds to the base coordinates x^i and the multiindex β corresponds to the fibre coordinates y^p. Let z^a be the local coordinates on $F(\mathbf{R}^n)$ and w^a_i be the induced coordinates on $TF(\mathbf{R}^n) \otimes \mathbf{R}^{m*}$. By the general theory, we are looking for $G^{r+1}_{m,n}$-equivariant maps

$$w^a_i = f^a_i(y^p_{i\alpha\beta}, z)\,.$$

The base homotheties imply a homogeneity condition

$$tf^a_i(y^p_{i\alpha\beta}, z) = f^a_i(t^{1+|\alpha|}y^p_{i\alpha\beta}, z)\ ,\ 0 \ne t \in \mathbf{R}\,.$$

By the homogeneous function theorem, f^a_i are linear in $y^p_{i\beta}$ and independent of $y^p_{i\alpha\beta}$ with $|\alpha| > 0$. Hence

$$w^a_i = f^{aj\beta}_{ip}(z)y^p_{j\beta}\ ,\ |\beta| = 0, \dots, r\,. \tag{7}$$

The original action of $G^1_{m,n}$ on $S^0 = (y^p_i)$ is of the form

$$\bar{y}^p_i = a^p_q y^q_j \tilde{a}^j_i + a^p_j \tilde{a}^j_i\,.$$

The induced action of $G^{r+1}_{m,n}$ on S^r is determined by the standard prolongation procedure. The kernel of the jet homomorphism $G^{r+1}_{m,n} \to G^r_{m,n}$ is an Abelian group, so that the subset $K^{r+1} \subset G^{r+1}_{m,n}$ defined by

$$a^i_j = \delta^i_j\ ,\ a^p_q = \delta^p_q\ ,\ a^p_{iq_1\dots q_r}\ \text{arbitrary, all other's zero} \tag{8}$$

is a subgroup. Each element of K^{r+1} is the $(r+1)$-jet of local isomorphism of $\mathbf{R}^{m,n}$, where restriction to the fiber $\{0\} \times \mathbf{R}^n \subset \mathbf{R}^m \times \mathbf{R}^n$ is the identity. Since V^F has order $(0, s, 0)$, the induced action on $F(\mathbf{R}^n)$ is also the identity. Hence z remain unchanged. By $a^i_j = \delta^i_j$, w^a_i remain unchanged too. Then the equivariancy of (7) with respect to K^{r+1} yields

$$f^{aj\beta}_{ip}(z)a^p_{j\beta} = 0 \ , \ |\beta| = r \ .$$

This implies $f^{aj\beta}_{ip} = 0$ for all $|\beta| = r$.

In the next step, we consider $K^r \subset G^r_{m,n}$ and the canonical homomorphism $\pi : G^{r+1}_{m,n} \to G^{r+1}_n$ determined by restricting the local isomorphisms of $\mathbf{R}^{m,n}$ preserving $(0,0)$ to the fiber $\{0\} \times \mathbf{R}^n$. We define $\tilde{K}^r \subset G^{r+1}_{m,n}$ to be the intersection of the kernel of π with the inverse image of K^r with respect to the jet projection $G^{r+1}_{m,n} \to G^r_{m,n}$. By equivariancy with respect to $\tilde{K}^r$, we obtain $f^{aj\beta}_{ip}(z) = 0$ for all $|\beta| = r-1$. In the last step of this backward procedure we find $f^{aj}_{ip}(z)a^p_j = 0$. Hence all $f's$ are zero, i.e. $\Delta = 0$. □

4. Finite order of natural operators

Now, we present a condition on F under which every natural operator D transforming connections on $Y \to M$ into connections on $V^F Y \to M$ has finite order.

Proposition 3. *If the standard fiber $F_0(\mathbf{R}^n)$ of F is compact or if $F_0(\mathbf{R}^n)$ contains a point z_o such that $F(b\,id_{\mathbf{R}^n})(z) \to z_o$ if $b \to 0$ for any $z \in F_0(\mathbf{R}^n)$, then every natural operator D transforming connections on $Y \to M$ into connections on $V^F Y \to M$ has finite order.*

Proof. This follows from the proof of Proposition 23.7 in [3], which can be generalized to our situation in the following way.

Consider the maps $\varphi_{a,b} : \mathbf{R}^{m+n} \to \mathbf{R}^{m+n}$, $\varphi_{a,b}(x,y) = (ax, by)$. Let us fix some $r \in \mathbf{N}$ and choose $a = b^{-r}$, $0 < b < 1$ arbitrary. Hence for every multiindex $\alpha = \alpha_1 + \alpha_2$, where α_1 includes all the derivatives with respect to the base coordinates while α_2 those with respect to the fibre coordinates, and for every connection $\Gamma = \Gamma^p_i(x,y)$ on $\mathbf{R}^{m,n}$

$$|\partial^{\alpha_1+\alpha_2}(\varphi^*_{a,b}\Gamma)(0,0)| = b^{r(1+|\alpha_1|)+1-|\alpha_2|}|\partial^{\alpha_1+\alpha_2}\Gamma(0,0)|$$

and so for all $|\alpha| \le r$ we get

$$|\partial^{\alpha_1+\alpha_2}(\varphi^*_{a,b}\Gamma)(0,0)| \le b|\partial^{\alpha}\Gamma(0,0)| \, .$$

On the other hand there is a compact subset $K \subset V^F_{(0,0)}(\mathbf{R}^{m,n}) = F_0(\mathbf{R}^n)$ ($K = F_0(\mathbf{R}^n)$ or K is a compact neighbourhood of z_o) such that for any $z \in V^F_{(0,0)}(\mathbf{R}^{m,n})$ $V^F\varphi_{a,b}(z) \in K$ for sufficiently small b. Hence Corollary 23.4 in [3] implies our assertion. □

Then we have the following corollary of Proposition 2.

Corollary 1. *If the standard fiber $F_0(\mathbf{R}^n)$ of F contains a point z_o such that $F(b\,id_{\mathbf{R}^n})(z) \to z_o$ if $b \to 0$ for any $z \in F_0(\mathbf{R}^n)$ or if $F_0(\mathbf{R}^n)$ is compact, then $\mathcal{V}^F$ is the only natural operator transforming connections on $Y \to M$ into connections on $V^F Y \to M$.*

The assumption of Proposition 3 or Corollary 1 is satisfied in the case F is a Weil functor T^A, [3], [5]. (In general, it is satisfied for all natural bundles which are the restrictions to $\mathcal{M}f_n$ of bundle functors on the whole category of manifolds with the point property, [3].) Then we have

Corollary 2. *If F is a Weil functor T^A, then $\mathcal{V}^F$ is the only natural operator transforming connections on $Y \to M$ into connections on $V^F Y \to M$.*

5. The case of vertical Weil bundles

Consider now a Weil bundle T^A in the role of F. Its vertical modification will be denoted by V^A. In this case, there is another way how to construct a connection on $V^A Y \to M$ from a connection $\Gamma : Y \to J^1 Y$. In [2], the first author constructed a natural identification

$$\kappa : V^A(J^1 Y \to M) \to J^1(V^A Y \to M) .$$

Clearly, V^A is a functor defined on the whole category $\mathcal{FM}$. If we apply it to the $\mathcal{FM}$-morphism Γ, we obtain a map $V^A\Gamma : V^A Y \to V^A(J^1 Y)$. Then

$$\kappa \circ V^A\Gamma : V^A \to J^1(V^A Y \to M) \tag{9}$$

is a connection on $V^A Y \to M$. The following corollary is an interesting application of Proposition 2 or Corollary 2.

Corollary 3. *Connection (9) coincides with $\mathcal{V}^A\Gamma$.*

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