Lenka Kadlčáková

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DIRAC OPERATOR IN CONTACT SYMPLECTIC PARABOLIC GEOMETRY

LENKA KADLČÁKOVÁ

ABSTRACT. In this paper, we find an analog of the standard (conformal) Dirac operator inside the contact symplectic parabolic geometry. The rôle of the group *Spin* and spinor representation is played by Metaplectic Lie group $Mp(n,\mathbb{R})$ and by its Segal-Shale-Weil representation. This representation is infinite dimensional, and so the Harish-Chandra category of finite K-modules must be introduced.

1. Parabolic Geometries

For reader's convenience, we start with a short summary concerning parabolic geometries and related staff, based on the representation theory.

Let $\mathfrak{g}$ be a $|k|$-graded (real, complex) semisimple Lie algebra, i.e.

$$\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k ,$$

such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$. We denote the subalgebra $\mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ of $\mathfrak{g}$ by $\mathfrak{p}$, and the subalgebra $\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ of $\mathfrak{p}$ by $\mathfrak{p}_+$. Let us define the *grading element* $E \in Z(\mathfrak{g}_0)$ to be the unique element in $\mathfrak{g}$ whose adjoint action is given by $[E, X] = jX$ for any $X \in \mathfrak{g}_j$ and any $j \in \{-k, \dots, k\}$. In particular, if $\mathfrak{g}$ is complex, there always exists a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ containing E, and one can choose a system Δ_+ of positive roots for $\mathfrak{h}$ in such a way that all root spaces corresponding to the simple roots are contained in $\mathfrak{g}_0 \oplus \mathfrak{g}_1$. The grading on $\mathfrak{g}$ is then given by the $\mathfrak{g}_1$-lengths of roots, i.e., if α is a root, then the root space $\mathfrak{g}_\alpha$ is contained in $\mathfrak{g}_i$, where i is the sum of all coefficients of simple roots in the expansion of α, with root spaces contained in $\mathfrak{g}_1$. This implies that $\mathfrak{p}$ is always the parabolic subalgebra of $\mathfrak{g}$ and the decomposition $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{p}_+$ is the Levi decomposition of $\mathfrak{p}$ onto its reductive and nilpotent part.

Note also, that for a $|k|$-graded real Lie algebra $\mathfrak{g}$, its complexification $\mathfrak{g}^{\mathbb{C}}$ is also $|k|$-graded, so in general we will deal with certain real forms of $(\mathfrak{g}, \mathfrak{p})$, where $\mathfrak{g}$ is complex semisimple Lie algebra and $\mathfrak{p}$ is its (complex) parabolic subalgebra.

We use the standard notation of crossed Dynkin diagrams in order to describe the pair $(\mathfrak{g}, \mathfrak{p})$. In particular, in the Dynkin diagram of $\mathfrak{g}$ we cross out the simple roots whose root spaces are contained in $\mathfrak{g}_1$.

Still take $\mathfrak{g}$ to be a $|k|$-graded Lie algebra over $\mathbb{R}, \mathbb{C}$ and let G be any Lie group with Lie algebra $\mathfrak{g}$. We define the subgroup $G_0 \subset G$ consisting of all $g \in G$, such that the adjoint action $Ad(g) : \mathfrak{g} \to \mathfrak{g}$ preserves the grading of $\mathfrak{g}$. Further, we denote by P

The paper is in final form and no version of it will be submitted elsewhere.

the subgroup of all $g \in G$ such that the adjoint action $Ad(g)$ preserves the filtration induced by grading of $\mathfrak{g}$, i.e. $Ad(g)(\mathfrak{g}_i) \subset \mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_k$, $i = -k, \ldots, k$. G_0 has Lie algebra $\mathfrak{g}_0$, P has Lie algebra $\mathfrak{p}$, and by definition G_0 is a subgroup of P. Note also, that the Killing form provides the identification $\mathfrak{g}_i^* = \mathfrak{g}_{-i}$, $i = -k, \ldots, k$, which is an isomorphism.

Definition 1.1. We define a (real) parabolic geometry of type (G, P) on a smooth manifold M, $\dim M = \dim G/P$, as the principal P-bundle $\mathcal{G} \xrightarrow{p} M$ equipped with a Cartan connection of type (G, P), i.e. with a differential form $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$, such that

$$\begin{aligned} &(1) \quad \omega(\zeta_X) = X \quad \text{for all} \quad X \in \mathfrak{p}\,, \\ &(2) \quad (r^b)^*\omega = Ad(b^{-1}) \circ \omega \quad \text{for all} \quad b \in P\,, \\ &(3) \quad \omega/_{T_u\mathcal{G}} \to \mathfrak{g} \quad \text{is a linear isomorphism for all} \quad u \in \mathcal{G}\,. \end{aligned}$$

In this definition, ζ_X denotes the fundamental vector field generated by $X \in \mathfrak{p}$ and r^b denotes the principal right action of $b \in P$. Consequently, ω gives a smooth P-equivariant trivialization of the tangent bundle $T\mathcal{G}$. Each $X \in \mathfrak{g}$ defines the constant vector field $\omega^{-1}(X)(u) = \omega_u^{-1}(X) \in T_u\mathcal{G}$

We call the (real, complex) generalized flag manifolds G/P the homogeneous (flat) model for parabolic geometry, the Cartan connection on G/P being the left invariant Maurer-Cartan form.

Let us consider a parabolic geometry $(\mathcal{G}, P)$ on a manifold M. We set the $\mathfrak{p}$-module structure on $\mathfrak{g}_-$ identifying it with $\mathfrak{g}/\mathfrak{p}$, and consider the map

$$\mathcal{G} \times \mathfrak{g}_- \to TM$$

defined by

$$(u, X) \to T_p \cdot \omega_u^{-1}(X)\,.$$

Due to P-equivariance of the Cartan connection, this map factors to a homomorphism

$$\mathcal{G} \times_P \mathfrak{g}_- \to TM\,,$$

which is clearly surjective. Comparing the dimensions of both spaces, one realizes, that this map is in fact a vector space isomorphism. Now, as we have an identification of P-modules $(\mathfrak{g}/\mathfrak{p})^*$ and $\mathfrak{p}_+$ provided by the Killing form on $\mathfrak{g}$, we use the same arguments as above and identify T^*M as the natural bundle associated to the P-module $\mathfrak{p}_+$, that is, $T^*M \simeq \mathcal{G} \times_P \mathfrak{p}_+$. Since we have the global parallelism ω, the filtration of the $\mathfrak{p}$-module $\mathfrak{g}$ transfers into the P-invariant filtration of $T\mathcal{G}$, i.e.

$$T\mathcal{G} = T^{-k}\mathcal{G} \supset \cdots \supset T^k\mathcal{G}\,.$$

Thus we have also the induced P-invariant filtration of the underlying tangent bundle TM, namely $T^iM = T_p(T^i\mathcal{G})$. We obtain immediately also the associated graded vector space

$$GrTM = (T^{-k}M/T^{-k+1}M) \oplus \cdots \oplus (T^{-2}M/T^{-1}M) \oplus T^{-1}M\,,$$

the parallelism of ω provides a reduction of the structure group of $GrTM$ to $G_0 \subset P$.

Definition 1.2. Let $\mathbb{E}$ be an irreducible $\mathfrak{p}$-module. For any Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ we define the operator

$$\nabla^\omega : C^\infty(\mathcal{G}, \mathbb{E})^P \to C^\infty(\mathcal{G}, \mathfrak{g}_- \otimes \mathbb{E})$$

given for all $X \in \mathfrak{g}_-, u \in \mathcal{G}, s \in C^\infty(\mathcal{G}, \mathbb{E})^P$ by

$$\nabla^\omega s(u)(X) = \mathcal{L}_{\omega^{-1}(X)} s(u) .$$

We call this operator the invariant derivative with respect to the Cartan connection ω.

Further, let us denote by $J^1\mathbb{E}$ the $\mathfrak{p}$-module $J^1\mathbb{E} = \mathbb{E} \oplus (\mathfrak{g}_-^* \otimes \mathbb{E})$. The action of $\mathfrak{p}$ on $J^1\mathbb{E}$ is given by

$$Z \cdot (v, \varphi) = \Big(\rho(Z)(v), \rho(Z) \circ \varphi - \varphi \circ ad_{\mathfrak{g}_-}(Z) + \rho(ad_{\mathfrak{p}}(Z)(X))(v)\Big)$$

for any $X \in \mathfrak{g}_-$.

As a result, any $\mathfrak{p}$-module homomorphism $\Psi : J^1\mathbb{E} \to \mathbb{F}$ for $\mathbb{F}$ an irreducible $\mathfrak{p}$ module, always provides an invariant differential operator; such operators are called strongly invariant. We are going to introduce an example of it, which will show up to be a formal symplectic analog of standard conformal Dirac operator.

2. Dirac Operator

The parabolic geometry of interest is represented by Dynkin diagram

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i.e. we have $|2|$-graded Lie algebra $\mathfrak{g} = sp(n+1, \mathbb{R}) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$, the real form of $C_{n+1} = sp(n+1, \mathbb{C})$. To determine the modules appearing in this gradation, we use the matrix notation introduced in [Yamaguchi]:

$\mathfrak{g}_0$	$\cdots \mathfrak{g}_1 \cdots$	$\mathfrak{g}_2$
$\vdots$ $\mathfrak{g}_{-1}$	$\mathfrak{g}_0^s \subset \mathfrak{g}_0$	$\vdots$ $\mathfrak{g}_1$
$\mathfrak{g}_{-2}$	$\cdots \mathfrak{g}_{-1} \cdots$	$\mathfrak{g}_0$

with

$$\begin{aligned}
\mathfrak{g}_0^s &= sp(n, \mathbb{R}), \quad \text{the real form of} \quad C_n = sp(n, \mathbb{C}) \\
\mathfrak{g}_0 &= sp(n, \mathbb{R}) \oplus \mathbb{R}E, \quad \text{where} \quad E \quad \text{is the grading element}, \\
\mathfrak{g}_1 &= \mathbb{R}^{2n}, \\
\mathfrak{g}_2 &= \mathbb{R},
\end{aligned}$$

we have

$$\begin{aligned}
\mathfrak{p} &= sp(n, \mathbb{R}) \oplus \mathbb{R}E \oplus \mathbb{R}^{2n} \oplus \mathbb{R} \\
\mathfrak{p}_+ &= \mathbb{R}^{2n} \oplus \mathbb{R} .
\end{aligned}$$

The homogeneous model for this parabolic geometry is the projective space $M = G/P = \mathbb{P}N; N = \{v \in \mathbb{R}^{2n+2}; \omega(v,v) = 0\}$ of null lines in symplectic vector space $(\mathbb{R}^{2n+2}, \omega)$. Further, the model for cotangent space T^*M is the direct sum $\mathfrak{g}_1 \oplus \mathfrak{g}_2 \simeq \mathbb{R}^{2n+1}$.

Note (see [Alekseevski, Slovák], [Yamaguchi]), that this is the only example of parabolic contact geometry for the Lie algebra C_n. This explains, why we have chosen the title "contact symplectic parabolic geometry".

The goal of this paper is to look for an analogy of the Dirac operator defined in orthogonal case for the parabolic geometry, which was just described. We have the following analogy:

$$\begin{array}{cc} Spin(n,\mathbb{R}) & Mp(n,\mathbb{R}) \\ \downarrow & \downarrow \\ SO(n,\mathbb{R}) & Sp(n,\mathbb{R}) \end{array}$$

$$Spin - \text{module } S, S^{\pm} \qquad Mp(n,\mathbb{R}) - \text{module } L, L_{\pm}$$

The rôle of $Spin(n)$ group is in the symplectic case played by the Metaplectic Lie group $Mp(n,\mathbb{R})$, which is the two-fold covering group of $Sp(n,\mathbb{R})$; however, it is not the universal covering. The representation L of Metaplectic Lie group on the Hilbert space $L^2(\mathbb{R}^n, e^{-|x|^2})$ of complex valued functions is called Segal-Shale-Weil (or metaplectic) representation. This representation is infinite dimensional, and we define it as follows:

Let us consider the following generators of $Sp(n,\mathbb{R})$:

$$\begin{aligned} g(a) &= \begin{pmatrix} a & 0 \\ 0 & (a^t)^{-1} \end{pmatrix} \quad \text{for } a \in GL(n,\mathbb{R}), \\ t(b) &= \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \quad \text{for } b = b^T \in GL(n,\mathbb{R}), \\ \beta &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \end{aligned}$$

Since $\{t(b); b \in S(n)\}$ is simply connected, $\widetilde{t(b)} = (t(b), \pm 1)$ can be viewed as an element in $Mp(n,\mathbb{R})$ so that $\widetilde{t(0)}$ is the identity of $Mp(n,\mathbb{R})$. For each $a \in GL(n,\mathbb{R})$ we take $(\det a)^{\frac{1}{2}}$ and identify $\widetilde{g(a)} = (g(a), (\det a)^{\frac{1}{2}})$ and $\widetilde{\beta} = (\beta, i^{\frac{1}{2}})$ as the elements in $Mp(n,\mathbb{R})$.

Definition 2.1. The metaplectic representation L on the space $L^2(\mathbb{R}^n, e^{-|x|^2})$ is defined on generators of $Mp(n,\mathbb{R})$ by

$$\begin{aligned} (L(\widetilde{g(a)})f)(x) &= (deta)^{\frac{1}{2}} f(a^t x), \\ (L(\widetilde{t(b)})f)(x) &= e^{-\frac{i}{2}(bx,x)} f(x), \\ (L(\widetilde{\beta})f)(x) &= \left(\frac{i}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} e^{i(x,y)} f(y)\, dy. \end{aligned} \tag{1}$$

Let us denote by K the maximal compact subgroup of $Sp(n,\mathbb{R})$, i.e. $K = U(n)$. Working with any infinite dimensional representation, it is necessary to proceed in the Harish-Chandra category of finite K-modules. The Segal-Shale-Weil representation L decomposes into two irreducible K-finite, infinite unitary representations of $sp(n,\mathbb{R})$, whose leading weights[1] are $(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})_K$ and $(\frac{3}{2}, \frac{1}{2}, \ldots, \frac{1}{2})_K$. We denote them L_+ and L_-.

The group $U(n) = U(1) \times SU(n)$ is reductive and we regard the representations L_+, L_- as the harmonic series of finite K-modules, written in the form

$$
\begin{aligned}
L_+ &\to (\varepsilon^{\frac{1}{2}}, (\{0\} + \{2\} + \{4\} + \ldots)), \\
L_- &\to (\varepsilon^{\frac{1}{2}}, (\{1\} + \{3\} + \{5\} + \ldots)),
\end{aligned}
\tag{2}
$$

where $\varepsilon^{\frac{1}{2}}$ is the square root of the determinant representation of $U(n)$ and $\{k\} = (k, 0, \ldots, 0)$ is the highest weight of the irreducible $SU(n)$-module. It can be regarded as a polynomial of degree k, these polynomials are embedded into the space $L^2(\mathbb{R}^n)$ with the norm $e^{-|x|^2}$ and their images form the orthogonal system in $L^2(\mathbb{R}^n)$ with respect to this measure.

As described in the introductory section, we hope to find contact symplectic Dirac operator inside the following scheme:

$$\Gamma(L^2(\mathbb{R}^n)) \to \Gamma(L^2(\mathbb{R}^n) \otimes T^*M) \to \Gamma(L^2(\mathbb{R}^n))$$

To do this, we need to decompose the tensor product $L^2(\mathbb{R}^n) \otimes (\mathbb{R}^{2n} \oplus \mathbb{R})$ into irreducible $sp(n, \mathbb{R})$-representations. In [Slovák, Souček], one finds the proof of the fact, that decomposing this tensor product, we can forget the trivial representation $\mathfrak{g}_2 = \mathbb{R}$ of $sp(n, \mathbb{R})$, in other words, every invariant operator of first order factors through $\mathfrak{g}_2$. The fact, that the target space of an invariant derivative in a parabolic geometry in question is $\mathfrak{g}_1 = \mathbb{R}^{2n}$ is the reason of the chosen name "contact symplectic Dirac operator".

What has to be done is to decompose the tensor product $L^2(\mathbb{R}^n) \otimes_{\mathbb{R}} \mathbb{R}^{2n} \simeq L^2(\mathbb{R}^n) \otimes_{\mathbb{C}} \mathbb{C}^{2n}$. In what follows, we shall describe the coordinate decomposition of $L^2(\mathbb{R}^n) \otimes_{\mathbb{C}} \mathbb{C}^{2n}$ onto the Kernel and Image invariant subspaces of symplectic Clifford multiplication.

Definition 2.2. The symplectic Clifford multiplication is the map

$$\mu : \mathbb{C}^{2n} \times L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$$

given by $\mu(v, f) = \tilde{\sigma}(v) f$. Here, $\tilde{\sigma}$ is complexified representation of the Heisenberg Lie algebra restricted to $\mathbb{C}^{2n} \hookrightarrow \mathbb{C}^{2n} \oplus \mathbb{C}$.

Let $\{e_1, \ldots, e_{2n}\}$ denote a symplectic basis of $\mathbb{R}^{2n}$, i.e. $\omega(e_i, e_{n+j}) = \delta_{i,j}$. We will denote by $\{e_1^*, \ldots, e_{2n}^*\}$ the dual basis of $(\mathbb{R}^{2n})^*$, and by $\{f_1, \ldots, f_{2n}\}$ the dual basis to $\{e_1, \ldots, e_{2n}\}$ with respect to the symplectic form ω, i.e. $\omega(e_i, f_j) = \delta_{i,j}$.

In particular, if $\{e_1, \ldots, e_{2n}\}$ is the canonical symplectic basis of $\mathbb{R}^{2n}$ then $\{f_1, \ldots, f_{2n}\}$ is given by the relations $f_i = e_{n+i}$, $f_{n+i} = -e_i$.

[1]By the expression 'leading weight' we mean the weight closest to the origin of the weight lattice corresponding to the irreducible $K = U(n)$ representation.

The space $L^2(\mathbb{R}^n) \otimes_{\mathbb{C}} \mathbb{C}^{2n}$ decomposes as the direct sum $V_1 \oplus V_2$, where

$$
\begin{aligned}
V_1 &= \{\sum_{i=1}^{2n} e_i s \otimes e_i^*;\ s_i \in L_\pm\} \subset L_\mp \otimes (\mathbb{C}^{2n})^* \\
V_2 &= \{\sum_{i=1}^{2n} s_i \otimes e_i^*;\ s_i \in L_\pm; \sum_{i=1}^{2n} f_i s_i = 0\} \subset L_\mp \otimes (\mathbb{C}^{2n})^*
\end{aligned} \tag{3}
$$

Lemma 2.3. *The symplectic Clifford multiplication is a map $L_\pm \to L_\mp$, in particular $\mu : \mathbb{C}^{2n} \times L_\pm \to L_\mp$.*

Proof. Fix the monomial $s_k \in L_\pm$. Also in further, it seems to be convenient to take as a base $\{e_1, \dots, e_{2n}\}$ of $\mathbb{C}^{2n}$ the vectors

$$
\begin{aligned}
e'_j &= (e_j + ie_{n+j}), \\
e'_{n+j} &= (e_j - ie_{n+j}),
\end{aligned}
$$

where $\{e_i\}_{i=1,\dots,2n}$ denotes the canonical symplectic base of $\mathbb{R}^{2n}$. For every $j = 1, \dots, 2n$,

$$
e'_j s_k = \sum_{i_1 \dots i_k} a_{i_1 \dots i_k} e'_j x_1^{i_1} x_2^{i_2} \dots x_k^{i_k} + a_{i_1 \dots i_k} e'_{n+j} x_1^{i_1} x_2^{i_2} \dots x_k^{i_k}\,. \tag{4}
$$

By definition of Clifford multiplication, the application of $\tilde{\sigma}$ reveals

$$
\begin{aligned}
(4) &= \sum_{i_1 \dots i_k} a_{i_1 \dots i_k} (x_i + i\frac{\partial}{\partial x_{n-i}}) x_1^{i_1} x_2^{i_2} \dots x_k^{i_k} \\
&\quad + a_{i_1 \dots i_k} (x_i - i\frac{\partial}{\partial x_{n-i}}) (x_1^{i_1} x_2^{i_2} \dots x_k^{i_k}),
\end{aligned}
$$

and we really end up in $L_\mp$. □

Theorem 2.4. *Let $\rho : Sp(n, \mathbb{R}) \to Aut(\mathbb{R}^{2n})$ denote the standard representation of the group $Sp(n, \mathbb{R})$, let ρ^* denote its dual representation, and let L denote the metaplectic representation. The spaces V_1 and V_2 defined above are, as the representation spaces of $(\rho^* \otimes L)$, $Sp(n, \mathbb{R})-$invariant subspaces. Moreover, the space V_1 is isomorphic as the $Sp(n, \mathbb{R})$-module to the $L_\mp$ part in the decomposition of the product $(L_\pm \otimes \mathbb{C}^{2n})$.*

Before proving this theorem, let us mention the fundamental property which can be found in [Howe].

Theorem 2.5. *Let $\pi : Mp(n, \mathbb{R}) \to Sp(n, \mathbb{R})$ denote the projection map. Then following relation holds:*

$$
L(m)\tilde{\sigma}(h)L(m^{-1}) = \tilde{\sigma}(\pi(m)(h)), \quad m \in Mp(n, \mathbb{R}), h \in \mathbb{C}^{2n}\,. \tag{5}
$$

Proof of Theorem (2.4). We are presenting the proof in such a form that it can be used at the same time without any change in orthogonal case.
The dual representation ρ^* acting on $(\mathbb{C}^{2n})^*$ is defined by

$$
\rho^*(m)v^*(v) = v^*(\rho(m^{-1})(v)), \forall v^* \in (\mathbb{C}^{2n})^*, \quad v \in \mathbb{C}^{2n}\,.
$$

Invariance of V_1 follows from the computation:

$$\begin{aligned}(L\otimes\rho^*)(m)\big[\sum_{j=1}^{2n} e_js\otimes e_j^*\big](v) &= \Big[\sum_{j=1}^{2n} L(m)(e_js)\otimes\rho^*(m)(e_j^*)\Big](v)\\ =\sum_j L(m)(e_js)e_j^*(\rho(m^{-1})(v)) &= L(m)(\Big[\sum_j e_j^*(\rho(m^{-1})(v))e_j\Big]\cdot s)\\ =L(m)\big[\rho(m^{-1})(v)\cdot s\big] &= v\cdot L(m)s\in V_1\,.\end{aligned}$$

And for V_2:

$$(L\otimes\rho^*)(m)(\sum_j s_j\otimes e_j^*) = \sum_j L(m)(s_j)\otimes\rho^*(m)(e_j^*) \tag{6}$$

In coordinates, $\rho^*(m)(e_j^*)=\sum_k\rho^*(m)(e_j^*)(e_k)\cdot e_k^*$, hence

$$\begin{aligned}(6) &= \sum_k\sum_j L(m)s_j\otimes\rho^*(m)(e_j^*)(e_k)\cdot e_k^*\\ &= \sum_k(\sum_j\rho^*(m)(e_j^*)(e_k)L(m)s_j)\otimes e_k^* = \sum_k s'_k\otimes e_k^*.\end{aligned}$$

To complete the proof we need to show that $\sum_k f_ks'_k=0$.

$$\begin{aligned}\sum_k f_ks'_k &= \sum_k f_k\sum_j\rho^*(m)(e_j^*)(e_k)L(m)s_j\\ &= \sum_k f_k\sum_j e_j^*(\rho(m^{-1})e_k)L(m)s_j\\ &= \sum_j\Big(\sum_k f_ke_j^*(\rho(m^{-1})e_k)\Big)L(m)s_j\\ &= \sum_j\Big(\sum_k\rho(m)(f_k)e_j^*(e_k)\Big)L(m)s_j\\ &= \sum_j\rho(m)f_jL(m)s_j = L(m)(\sum_j f_js_j)=0\,.\end{aligned}$$

□

We can now write down the explicit forms of the projections onto V_1 and V_2. Take $\sum_k s_k\otimes e_k^*\in L^2(\mathbb{R}^n)\otimes(\mathbb{C}^{2n})^*$, then

$$\begin{aligned}\pi_{V_1}: L_\pm\otimes(\mathbb{C}^{2n})^* &\to V_1\\ \pi_{V_1}(\sum_k s_k\otimes e_k^*) &= \frac{1}{n}\sum_k[e_k(\sum_j f_js_j)]\otimes e_k^* \qquad (7)\\ \pi_{V_2}: L_\pm\otimes(\mathbb{C}^{2n})^* &\to V_2\\ \pi_{V_2}(\sum_k s_k\otimes e_k^*) &= \sum_k[s_k-\frac{1}{n}e_k(\sum_j f_js_j)]\otimes e_k^* \qquad (8)\end{aligned}$$

And finally, let us choose any Weyl connection ∇^w in the given parabolic geometry, let $j: L\to V_1\subset L\otimes(\mathbb{C}^{2n})^*$ be the embedding given in any frame $\{e_i\}$ for $\mathbb{R}^{2n}$ by

$s \to \frac{1}{n}\sum_{i=1}^{2n} e_j \cdot s \otimes e_j^*; s \in L$. We define the contact symplectic Dirac operator as the composition

$$D = j^{-1} \circ \pi_{V_1} \circ \nabla^w : \Gamma(L_\pm) \to \Gamma(L_\pm \otimes \mathbb{C}^{2n}) \to \Gamma(L_\mp).$$

3. Invariance of Symplectic Dirac Operator

To verify the invariance of just defined Dirac operator, some more computations have to be done. For finite-dimensional irreducible representations of any Lie algebra $\mathfrak{g}$, we have the explicit way to classify all invariant first order differential operators by computing corresponding Casimir elements (see [Slovák, Souček]). The situation in infinite-dimensional case is slightly more complicated. There is a chance, however, that we can reformulate the statements used for finite dimensional representations and get some results also in this case. Let us start with some definitions and basic facts.

Definition 3.1. Let $\{X_i\}_{i=1}^n$ be a basis of a semisimple Lie algebra $\mathfrak{g}$, denote its Killing form by B, take $g_{ij} = B(X_i, X_j)$ and $g^{ij} = (g_{ij})^{-1}$. Then the element

$$\mathfrak{c} \;=\; \sum_{i,j=1}^{n} g^{ij} X_i X_j \tag{9}$$

of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ of $\mathfrak{g}$ is called Casimir element.

Remark 3.2. Let Δ denote the set of all roots $\alpha \in \mathfrak{h}^*$ for $\mathfrak{g}$, let Δ_+ denote the set of all positive roots, and let S denote the set of all simple roots for $\mathfrak{g}$. We can rewrite the Casimir element to the form

$$\mathfrak{c} \;=\; \sum_{\alpha\in S} h_\alpha \tilde{h}_\alpha + \sum_{\alpha\in\Delta_+} (X_\alpha Z_\alpha + Z_\alpha X_\alpha). \tag{10}$$

Here, $\{h_\alpha\}, \{\tilde{h}_\alpha\}$ are the dual (with respect to the Killing form) basis for $\mathfrak{h}$, and X_α, Z_α are the dual generators of $\mathfrak{g}_\alpha$, $\mathfrak{g}_{-\alpha}$.

Remark 3.3. The Casimir element $\mathfrak{c}$ of $\mathfrak{U}(\mathfrak{g})$ is independent of the choice of basis $\{X_i\}$, and it belongs to the center of $\mathfrak{U}(\mathfrak{g})$. Moreover, $\mathfrak{c}$ operates by the scalar $(\lambda, \lambda+2\rho)$ on a finite dimensional, irreducible representation V_λ of the (complex, real) semisimple Lie group $\mathfrak{g}$, and this scalar value is nonzero if the representation is non-trivial.

The following theorem shows, that the Casimir element acts as the multiple of identity also on any irreducible, infinite dimensional unitary module, [Sugiura].

Theorem 3.4. *Let U be an irreducible unitary representation of the (real) Lie group G on a Hilbert space H, let $\mathfrak{c}$ be the Casimir element of G and put $\Omega = U'(\mathfrak{c})$, where U' denotes the differential representation of U. Then Ω acts by multiple of identity operator on H_∞, the space of C^∞-vectors for U. That is, there exists a real number q such that*

$$\Omega x = qx \quad \textit{for any} \quad x \in H_\infty.$$

Before we formulate the classification theorem of the invariant first order differential operators, let us denote:

$\mathfrak{c}$ ... the quadratic Casimir element of $\mathfrak{g}_0$,
$\mathfrak{c}_S$... the quadratic Casimir element of $\mathfrak{g}_0^S$,
$\mathfrak{c}_E$... the quadratic Casimir corresponding to the action of grading element E.

Definition 3.5. Let E be the grading element in $\mathfrak{g}_0$, let $\mathbb{E}$ be the irreducible $\mathfrak{g}_0^S$-representation, choose the Killing form B normalized in such a way that $(E,E)_B = 1$. We define the generalized conformal weight w of E by the equation

$$Ev = w.v \quad \text{for any smooth} \quad v \in \mathbb{E}.$$

Theorem 3.6. *Let $\mathfrak{g}$ be a (real) graded Lie algebra and $\mathfrak{g}^{\mathbb{C}}$ its (graded) complexification ($\mathfrak{g}_i = \mathfrak{g} \cap \mathfrak{g}_i^{\mathbb{C}}$). Let $\mathbb{E}(w)$ be a complex, unitary, irreducible, infinite-dimensional representation of $\mathfrak{g}_0^S$ with the generalized conformal weight w and let $\mathfrak{g}_1^{\mathbb{C}} = \sum_j \mathfrak{g}_1^j$ be the decomposition of $\mathfrak{g}^{\mathbb{C}}$ into irreducible $\mathfrak{g}_0^S$-submodules. Further, suppose that we have the decomposition*

$$\mathfrak{g}_1 \otimes_{\mathbb{R}} \mathbb{E}(w) = \mathfrak{g}_1^{\mathbb{C}} \otimes_{\mathbb{C}} \mathbb{E}(w) = \sum_j \mathfrak{g}_1^j \otimes_{\mathbb{C}} \mathbb{E}(w) = \sum_{j,l} \mathbb{E}_{j,l}(w+1)$$

onto unitary irreducible $\mathfrak{g}_0^S$-modules. Let $\pi_{j,l}$ denote the corresponding projections. Then the differential operator $D_{j,l} := \pi_{j,l} \circ \nabla$ with the generalized conformal weight w is an invariant differential operator if and only if

$$0 = \mathfrak{c}(\mathbb{E}_{j,l}(w+1)) - \mathfrak{c}(\mathbb{E}(w)) - \mathfrak{c}(\mathfrak{g}_1^j) \tag{11}$$

Rewriting the last equation (11) for each j, l,

$$\begin{aligned} 0 &= \mathfrak{c}_S(\mathbb{E}_{j,l}(w+1)) - \mathfrak{c}_S(\mathbb{E}(w)) - \mathfrak{c}_S(\mathfrak{g}_1^j) + \\ &+ \underbrace{\mathfrak{c}_E(\mathbb{E}_{j,l}(w+1))}_{=(w+1)^2} - \underbrace{\mathfrak{c}_E(\mathbb{E}(w))}_{=w^2} - \underbrace{\mathfrak{c}_E(\mathfrak{g}_1^j)}_{=1} \end{aligned} \tag{12}$$

we get the explicit formula for the generalized conformal weight of $D_{j,l}$, namely

$$w = \frac{1}{2}\Big(\mathfrak{c}_S(\mathbb{E}(w)) - \mathfrak{c}_S(\mathbb{E}_{j,l}(w+1)) + \mathfrak{c}(\mathfrak{g}_1^j)\Big).$$

Proof. See [Slovák, Souček], nothing changes using infinite dimensional representations. A basic difference from the finite dimensional case is, that we do not always have the decomposition of $\mathfrak{g}_1^{\mathbb{C}} \otimes_{\mathbb{C}} \mathbb{E}(w)$ onto irreducible $\mathfrak{g}_0^S$ modules. □

In our case of parabolic contact symplectic geometry, we have:

$$\begin{aligned} \mathbb{E}(w) &= L_{\pm} \\ \mathbb{E}_j(w+1) &= L_{\mp} \\ \mathfrak{g}_1^{\mathbb{C}} &= \mathbb{C}^{2n} \text{ irreducible } 2n \text{ dimensional (complexified)} \\ & \qquad \text{representation of } sp(n,\mathbb{R}). \end{aligned}$$

Explicit computations give the equality

$$\mathfrak{c}(L_+) = \mathfrak{c}(L_-),$$

which is analogical to the orthogonal case. Other computational exercise gives

$$\begin{aligned} \mathfrak{c}(\mathfrak{g}_1^{\mathbb{C}}) &= \frac{-2n^2-n}{4(4n+4)}, \\ w &= \frac{(1+2n)(2+n)}{2(n+1)}. \end{aligned}$$

So we have proved the following:

Theorem 3.7. *The contact symplectic Dirac operator*

$$D : \Gamma(L^2(\mathbb{R}^n)) \to \Gamma(L^2(\mathbb{R}^n) \otimes \mathbb{C}^{2n}) \to \Gamma(L^2(\mathbb{R}^n))$$

is invariant differential operator of the first order for contact symplectic parabolic invariant theory and for the generalized conformal weight $w = \frac{(1+2n)(2+n)}{2(n+1)}$.

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MATHEMATICAL INSTITUTE OF CHARLES UNIVERSITY
SOKOLOVSKÁ 83, 186 75 PRAGUE
CZECH REPUBLIC
E-mail: kadlcak@karlin.mff.cuni.cz