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MORE ON DEFORMED OSCILLATOR ALGEBRAS AND EXTENDED UMBRAL CALCULUS

A.K. KWAŚNIEWSKI, E. GRĄDZKA

ABSTRACT. $\psi(q)$ -calculus is an almost unavoidable extension of finite operator calculus of Rota [1]. Main results of Rotas' finite operator calculus might be quite easily given their ψ -extensions. The specific $\psi_n(q) = [n_q!]^{-1}$ case is known to be relevant for quantum groups investigation [2]–[5]. In general $\psi(q)$ -calculus is as a matter of fact Ward's "*calculus of sequences*" [6] in Rotas' finite operator calculus form [7]. This we owe to Viskov and other distinguished authors (see for example [8]–[13]). Here we show that such $\psi(q)$ -umbral calculus leads to infinitely many new ψ -deformed "quantum-like" oscillator algebras representations. Among others one may formulate q -extended finite operator calculus with help of the "quantum q -plane" q -commuting variables $A, B : AB - qBA \equiv [A, B]_q = 0$ as done in [11], [12]. This presentation is mostly an editorial actualization and enrichment of [14] based on [15] (see also [1]) and is intended to be further extension of last years talks given at Srní.

1. FEW BASIC NOTIONS OF $\psi(q)$ -EXTENDED UMBRAL CALCULUS

$\psi(q)$ -extended umbral calculus is arrived at [8], [9] by considering not only polynomial sequences of binomial type but also of $\{s_n\}_{n \geq 1}$ -binomial type where $\{s_n\}_{n \geq 1}$ -binomial coefficients are defined with help of the generalized factorial $n_S! = s_1 s_2 s_3 \cdots s_n$; $S = \{s_n\}_{n \geq 1}$ is an arbitrary sequence with the condition $s_n \neq 0, n \in N$. Then the extension relies on the notion of ∂_ψ -shift invariance of ∂_ψ -delta operators. Here the linear operator ∂_ψ acting on the algebra of polynomials denotes the ψ -derivative i.e. $\partial_\psi x^n = n_\psi x^{n-1}$; $n \geq 0$ and n_ψ denotes the ψ -deformed number (see also [6] and [13]) where in conformity with Viskov notation we put

$$\begin{aligned} n_\psi &\equiv \psi_{n-1}(q) \psi_n^{-1}(q) \text{ hence } (0_\psi! = 1) \\ n_\psi! &\equiv \psi_n^{-1}(q) \equiv n_\psi(n-1)_\psi(n-2)_\psi(n-3)_\psi \cdots 2_\psi 1_\psi \text{ and} \\ n_\psi^k &= n_\psi(n-1)_\psi \cdots (n-k+1)_\psi. \end{aligned}$$

We choose to work with \mathfrak{S} — the family of functions sequences such that:
 $\mathfrak{S} = \{\psi : R \supset [a, b]; q \in [a, b]; \psi(q) : Z \rightarrow F; \psi_0(q) = 1; \psi_n(q) \neq 0; \psi_{-n}(q) = 0; n \in N\}$. With the choice $\psi_n(q) = [R(q^n)!]^{-1}$ and $R(x) = \frac{1-x}{1-q}$ we get the well known

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q -factorial $n_q! = n_q(n-1)_q!$; $1_q! = 0_q! = 1$ while the ψ -derivative ∂_ψ becomes now the Jackson's derivative (see [16])

$$\partial_q : (\partial_q \varphi)(x) = \frac{\varphi(x) - \varphi(qx)}{(1-q)x}.$$

A polynomial sequence $\{p_n\}_0^\infty$ is called to be of ψ -binomial type if it satisfies the recurrence

$$E^\psi(\partial_\psi)p_n(x) \equiv p_n(x +_\psi y) \equiv \sum_{k=0}^n \binom{n}{k}_\psi p_k(x) p_{n-k}(y); \text{ where } \binom{n}{k}_\psi \equiv \frac{n!}{k! \psi^{k(n-k)}}.$$

$E^\psi(\partial_\psi) \equiv \exp_\psi\{y\partial_\psi\} = \sum_{k=0}^\infty \frac{y^k \partial_\psi^k}{k! \psi^k}$ denotes a generalized translation operator [6] and ∂_ψ -shift invariance is defined accordingly. The algebra Σ_ψ is the algebra of all F -linear ∂_ψ -shift invariant operators T acting on the algebra P of polynomials. We assume that $\text{char } F = 0$ for any field F chosen. In another words

$$\forall \alpha \in F \quad [T, E^\alpha(\partial_\psi)] = 0; \text{ char } F = 0.$$

One then introduces the notion of ∂_ψ -delta operator according to Definition 1.1.

Definition 1.1. Let $Q(\partial_\psi) : P \rightarrow P$; the linear operator $Q(\partial_\psi)$ is a ∂_ψ -delta operator iff

- (1) $Q(\partial_\psi)$ is ∂_ψ -shift invariant;
- (2) $Q(\partial_\psi)(\text{id}) = \text{const} \neq 0$.

As in unextended case [7] — one may construct [1] the bijective correspondence between ∂_ψ -delta operators with their ∂_ψ -basic polynomial sequences.

Definition 1.2. Let $Q(\partial_\psi) : P \rightarrow P$ be the ∂_ψ -delta operator. A polynomial sequence $\{p_n\}_{n \geq 0}$; $\deg p_n = n$ such that:

- (1) $p_0(x) = 1$;
- (2) $p_n(0) = 0$; $n > 0$;
- (3) $Q(\partial_\psi)p_n = n_\psi p_{n-1}$ is called the ∂_ψ -basic polynomial sequence of the ∂_ψ -delta operator $Q(\partial_\psi)$.

Now using the fact that $\forall Q(\partial_\psi) \exists!$ invertible $S_{\partial_\psi} \in \Sigma_\psi$ such that $Q(\partial_\psi) = \partial_\psi S_{\partial_\psi}$ one may prove (analogously to special cases [7], [12]) the crucial Theorem 1.1 (see [1], [10]).

Theorem 1.1. Let $\{p_n(x)\}_{n=0}^\infty$ be ∂_ψ -basic polynomial sequence of the ∂_ψ -delta operator $Q(\partial_\psi)$:

$$Q(\partial_\psi) = \partial_\psi S_{\partial_\psi}. \text{ Then for } n > 0:$$

- (1) $p_n(x) = Q(\partial_\psi)' S_{\partial_\psi}^{-n-1} x^n$;
- (2) $p_n(x) = S_{\partial_\psi}^{-n} x^n - \frac{n_\psi}{n} (S_{\partial_\psi}^{-n})' x^{n-1}$;
- (3) $p_n(x) = \frac{n_\psi}{n} \hat{x}_\psi S_{\partial_\psi}^{-n} x^{n-1}$;
- (4) $p_n(x) = \frac{n_\psi}{n} \hat{x}_\psi (Q(\partial_\psi)')^{-1} p_{n-1}(x)$.

In order to prove this one uses the properties of the Pincherle ψ -derivative.

Definition 1.3. The Pincherle ψ -derivative i.e. the linear map $' : \Sigma_\psi \rightarrow \Sigma_\psi$;

$$T' = T \hat{x}_\psi - \hat{x}_\psi T \equiv [T, \hat{x}_\psi]$$

where the linear map $\hat{x}_\psi : P \rightarrow P$; is defined in the basis $\{x^n\}_{n \geq 0}$ as follows

$$\hat{x}_\psi x^n = \frac{\psi_{n+1}(q)(n+1)}{\psi_n(q)} x^{n+1} = \frac{(n+1)}{(n+1)_\psi} x^{n+1}; \quad n \geq 0.$$

One may also define Sheffer ∂_ψ -polynomials which constitute the more general class of polynomial sequences than the class of ∂_ψ -basic polynomial sequences.

Definition 1.4. A polynomial sequence $\{s_n(x)\}_{n=0}^\infty$ is called the sequence of Sheffer ∂_ψ -polynomials of the ∂_ψ -delta operator $Q(\partial_\psi)$ iff

- (1) $s_0(x) = c \neq 0$;
- (2) $Q(\partial_\psi) s_n(x) = n_\psi s_{n-1}(x)$.

The following proposition relates Sheffer ∂_ψ -polynomials of the ∂_ψ -delta operator $Q(\partial_\psi)$ to the unique ∂_ψ -basic polynomial sequence of the ∂_ψ -delta operator $Q(\partial_\psi)$:

Proposition 1.1. Let $Q(\partial_\psi)$ be a ∂_ψ -delta operator with ∂_ψ -basic polynomial sequence $\{q_n(x)\}_{n=0}^\infty$. Then $\{s_n(x)\}_{n=0}^\infty$ is a sequence of Sheffer q -polynomials of the ∂_ψ -delta operator $Q(\partial_\psi)$ iff there exists a ∂_ψ -shift invariant operator S_{∂_ψ} such that $s_n(x) = S_{\partial_\psi}^{-1} q_n(x)$.

Examples: According to Proposition 1.1 with $Q(\partial_q) = \partial_q$ and $S = \exp_\psi\{\frac{1}{2}\alpha\partial_q^2\}$ we get q -Hermite polynomials while with choice $Q(\partial_q) = \frac{\partial_q}{\partial_q - 1}$ and $S = (1 - \partial_q)^{-\alpha-1}$ we obtain q -Laguerre polynomials $L_{n,q}^{(\alpha)}(x)$ of order α . ψ -extensions include of course q -Hermite, q -Laguerre polynomials $L_{n,q}^{(\alpha)}(x)$ of order α with their ψ -correspondents. These are already well known q -Sheffer polynomials [17], [11], [12]. Specifically q -Laguerre polynomials $L_{n,q}^{(-1)}(x) \equiv L_{n,q}(x)$ form the ∂_q -basic polynomial sequence $\{L_{n,q}(x)\}_{n \geq 0}$ of the ∂_q operator $Q(\partial_q) = -\sum_{k=0}^\infty \partial_q^{k+1} \equiv \frac{\partial_q}{\partial_q - 1} \equiv -[\partial_q + \partial_q^2 + \partial_q^3 + \partial_q^4 + \partial_q^5 + \dots]$. Using then Theorem 1.1 one arrives at the explicit form of $L_{n,q}(x)$. Namely:

$$\begin{aligned} L_{n,q}(x) &= \frac{n_q}{n} \hat{x}_q \left[\frac{1}{\partial_q - 1} \right]^{-n} x^{n-1} = \frac{n_q}{n} \hat{x}_q (\partial_q - 1)^n x^{n-1} = \\ &= \frac{n_q}{n} \hat{x}_q \sum_{k=1}^n (-1)^k \binom{n}{k}_q \partial_q^{n-k} x^{n-1} = \frac{n_q}{n} \sum_{k=1}^n (-1)^k \binom{n}{k}_q (n-1)_{\frac{n-k}{q}} \frac{k}{k_q} x^k = \\ &= \frac{n_q}{n} \sum_{k=1}^n (-1)^k \frac{n_q!}{k_q!} \frac{(n-1)_{\frac{n-k}{q}}}{(n-k)_q!} \frac{k}{k_q} x^k. \end{aligned}$$

So finally

$$(1.1) \quad L_{n,q}(x) = \frac{n_q}{n} \sum_{k=1}^n (-1)^k \frac{n_q!}{k_q!} \binom{n-1}{k-1}_q \frac{k}{k_q} x^k.$$

Note: ψ -extended case is covered in this example just by replacement $q \rightarrow \psi$.

With the choice $\psi_n(q) = [R(q^n)!]^{-1}$ we arrive at interesting R -Laguerre polynomials.

Let us also stress here again that q -deformed quantum oscillator algebra provides a natural setting for q -Laguerre polynomials and q -Hermite polynomials [18], [19], [20].

$sl_q(2)$ and the q -oscillator algebra give rise to basic geometric functions as matrix elements of certain operators in analogy with Lie theory [18], [19]. Also automorphisms of the q -oscillator algebra lead to Sheffer q -polynomials for example to q -generalization of the Charlier polynomials [18], [19].

2. EXTENDED UMBRAL CALCULUS AND ψ -DEFORMED "QUANTUM OSCILLATOR" ALGEBRAS

∂_q -delta operators and their duals and similarly ∂_ψ -delta operators with their duals provide us with pairs of generators of ψ -deformed quantum oscillator-like algebras (see Remark 2.2). Namely as we shall see: $[Q(\partial_\psi), \hat{x}_{Q(\partial_\psi)}]_{\hat{q}_{\psi,Q}} = \text{id}$. With the choice

$\psi_n(q) = [R(q^n)!]^{-1}$ and $R(x) = \frac{1-x}{1-q}$ we get the well known q -deformed oscillator dual pair of operators — generators of the well known q -Heisenberg-Weyl algebra. These oscillator-like algebras generators and q -oscillator-like algebras generators are encountered explicitly or implicitly in [2], [3] and in many other subsequent references — see [26], [5] and references therein. In many such references [18], [19] q -Laguerre and q -Hermite or q -Charlier polynomials appear which are just either Sheffer ψ -polynomials or just ∂_ψ -basic polynomial sequences of the ∂_ψ -delta operators $Q(\partial_\psi)$ for $\psi_n(q) = \frac{1}{R(q^n)!}$; $R(x) = \frac{1-x}{1-q}$ and corresponding choice of $Q(\partial_\psi)$ functions of ∂_ψ (for example $Q = \text{id}$). The case $\psi_n(q) = \frac{1}{R(q^n)!}$; $n_\psi = n_R$; $\partial_\psi = \partial_R$ and $n_{\psi(q)} = n_{R(q)} = R(q^n)$ appears implicitly in [21] where advanced theory of general quantum coherent states is being developed. However there is no mention of $R(q^n)$ -umbral calculus in [21] neither in " q -references" quoted in this note. In the q -case it was noticed among others also in [22] that commutation relations for the q -oscillator-like algebras generators from [2, 3] and others (see [5]) might be chosen in appropriate operator variables to be of the form [22]:

$$(2.1) \quad AA^+ - \mu A^+A = 1; \quad \mu = q^2$$

As for the Fock space representation of normalized eigenstates $|n\rangle$ of excitation number operator N various q -deformations of the natural number n are used in literature on quantum groups and at least some families of quantum groups may be constructed from q -analogues of Heisenberg algebra [2], [3], [22], [4]. Our q -oscillator algebras generators are just the ∂_q -delta operators $Q(\partial_q)$ and their duals i.e. basic objects of the q -extended finite operator calculus of Rota. (An elementary example: $\partial_q \hat{x} - q \hat{x} \partial_q = \text{id}$.)

Here in below we shall propose a ψ -extension of the q -oscillator model algebra using basic concepts of Viskov's ψ -extension of calculus of Rota.

Definition 2.1. Let $\{p_n\}_{n \geq 0}$ be the ∂_q -basic polynomial sequence of the ∂_q -delta operator $Q(\partial_q)$. A linear map $\hat{x}_{Q(\partial_q)} : P \rightarrow P$; $\hat{x}_{Q(\partial_q)} p_n = p_{n+1}$; $n \geq 0$ is called the operator dual to $Q(\partial_q)$.

For $Q = \text{id}$ we have : $\hat{x}_{Q(\partial_q)} \equiv \hat{x}_{\partial_q} \equiv \hat{x}$.

Definition 2.2. Let $\{p_n\}_{n \geq 0}$ be the ∂_ψ -basic polynomial sequence of the ∂_ψ -delta operator $Q(\partial_\psi) = Q$. Then the $\hat{q}_{\psi,Q}$ -operator is a liner map;

$$\hat{q}_{\psi,Q} : P \rightarrow P; \quad \hat{q}_{\psi,Q} p_n = \frac{(n+1)_\psi - 1}{n_\psi} p_n; \quad n \geq 0.$$

We call the $\hat{q}_{\psi,Q}$ operator the $\hat{q}_{\psi,Q}$ -mutator operator.

Note: For $Q = \text{id}$ $Q(\partial_\psi) = \partial_\psi$ the natural notation is $\hat{q}_{\psi,\text{id}} \equiv \hat{q}_\psi$. For $Q = \text{id}$ and $\psi_n(q) = \frac{1}{R(q^n)!}$ and $R(x) = \frac{1-x}{1-q}$ $\hat{q}_{\psi,Q} \equiv \hat{q}_{R,\text{id}} \equiv \hat{q}_R \equiv \hat{q}_{q,\text{id}} \equiv \hat{q}_q \equiv \hat{q}$ and $\hat{q}_{\psi,Q}x^n = qx^n$.

Definition 2.3. Let A and B be linear operators acting on P ;
 $A : P \rightarrow P$; $B : P \rightarrow P$. Then $AB - \hat{q}_{\psi,Q}BA \equiv [A, B]_{\hat{q}_{\psi,Q}}$ is called $\hat{q}_{\psi,Q}$ -mutator of A and B operators.

Note: $Q(\partial_\psi) \hat{x}_{Q(\partial_\psi)} - \hat{q}_{\psi,Q} \hat{x}_{Q(\partial_\psi)} Q(\partial_\psi) \equiv [Q(\partial_\psi), \hat{x}_{Q(\partial_\psi)}]_{\hat{q}_{\psi,Q}} = \text{id}$.

This is easily verified in the ∂_ψ -basic $\{p_n\}_{n \geq 0}$ of the ∂_ψ -delta operator $Q(\partial_\psi)$.

Equipped with pair of operators $(Q(\partial_\psi), \hat{x}_{Q(\partial_\psi)})$ and $\hat{q}_{\psi,Q}$ -mutator we have at our disposal all possible representants of "canonical pairs" of differential operators on the P algebra. For historical reasons let us however at first quote a suitable remark [1].

Remark 2.1. The ψ -derivative is a particular example of a linear operator that reduces by one the degree of any polynomial. In 1901 it was proved [23] by Pincherle and Amaldi that every linear operator T mapping P into P may be represented as infinite series in operators \hat{x} and D . In 1986 Kurbanov and Maximov [24] supplied the explicit expression for such series in most general case of polynomials in one variable; namely according to Proposition 1 from [24] one has: "Let \mathcal{D} be a linear operator that reduces by one each polynomial. Let $\{q_n(\hat{x})\}_{n \geq 0}$ be an arbitrary sequence of polynomials in the operator \hat{x} . Then $T = \sum_{n \geq 0} q_n(\hat{x}) \mathcal{D}^n$ defines a linear operator that maps polynomials into polynomials. Conversely, if T is linear operator that maps polynomials into polynomials then there exists a unique expansion of the form

$$T = \sum_{n \geq 0} q_n(\hat{x}) \mathcal{D}^n.$$

Note: In 1996 this was extended to algebra of many variables polynomials [25].

Remark 2.2. The importance of the pair of dual operators: $Q(\partial_\psi)$ and $\hat{x}_{Q(\partial_\psi)}$ is reflected by the facts:

- $Q(\partial_\psi) \hat{x}_{Q(\partial_\psi)} - \hat{q}_{\psi,Q} \hat{x}_{Q(\partial_\psi)} Q(\partial_\psi) \equiv [Q(\partial_\psi), \hat{x}_{Q(\partial_\psi)}]_{\hat{q}_{\psi,Q}} = \text{id}$.
- Let $\{q_n(\hat{x}_{Q(\partial_\psi)})\}_{n \geq 0}$ be an arbitrary sequence of polynomials in the operator $\hat{x}_{Q(\partial_\psi)}$. Then $T = \sum_{n \geq 0} q_n(\hat{x}_{Q(\partial_\psi)}) Q(\partial_\psi)^n$ defines a linear operator that maps polynomials into polynomials. Conversely, if T is linear operator that maps polynomials into polynomials then there exists a unique expansion of the form

$$(2.2) \quad T = \sum_{n \geq 0} q_n(\hat{x}_{Q(\partial_\psi)}) Q(\partial_\psi)^n.$$

Equipped with pair of operators $(Q(\partial_\psi), \hat{x}_{Q(\partial_\psi)})$ and $\hat{q}_{\psi,Q}$ -mutator we have at our disposal all possible representants of "canonical pairs" of differential operators on the P algebra such that:

- the above unique expansion $T = \sum_{n \geq 0} q_n(\hat{x}_{Q(\partial_\psi)}) Q(\partial_\psi)^n$ holds

- b) we have the structure of ψ -umbral or ψ -extended finite operator calculus - coworking.

3. LOOKING FOR ψ -ANALOGUE OF QUANTUM q -PLANE FORMULATION

Cigler and Kirschenhofer defined in [11, 12] the polynomial sequence $\{p_n\}_0^\infty$ of q -binomial type equivalently by

$$(3.1) \quad p_n(A+B) \equiv \sum_{k \geq 0} \binom{n}{k}_q p_k(A) p_{n-k}(B) \quad \text{where } [B, A]_q \equiv BA - qAB = 0.$$

A and B might be interpreted then as coordinates on quantum q -plane. For example $A = \hat{x}$ and $B = y\hat{Q}$ where $\hat{Q}\varphi(x) = \varphi(qx)$. With this being adopted the following identification holds:

$$p_n(x + {}_q y) \equiv E^y(\partial_q) p_n(x) = \sum_{k \geq 0} \binom{n}{k}_q p_k(x) p_{n-k}(y) = p_n(\hat{x} + y\hat{Q}) 1$$

Also q -Sheffer polynomials $\{s_n(x)\}_{n=0}^\infty$ are defined equivalently (see 2.1.1. Kirschenhofer in [12]) by

$$(3.2) \quad s_n(A+B) \equiv \sum_{k \geq 0} \binom{n}{k}_q s_k(A) p_{n-k}(B)$$

where $[B, A]_q \equiv BA - qAB = 0$ and $\{p_n(x)\}_{n=0}^\infty$ of q -binomial type. For example $A = \hat{x}$ and $B = y\hat{Q}$ where $\hat{Q}\varphi(x) = \varphi(qx)$. Then the following identification takes place:

$$(3.3) \quad s_n(x + {}_q y) \equiv E^y(\partial_q) s_n(x) = \sum_{k \geq 0} \binom{n}{k}_q s_k(x) p_{n-k}(y) = s_n(\hat{x} + y\hat{Q}) 1$$

This means that one may formulate q -extended finite operator calculus with help of the "quantum q -plane" q -commuting variables A, B :

$$AB - qBA \equiv [A, B]_q = 0.$$

Let us now try to formulate — perhaps in vain — the basic notions of ψ -extended finite operator calculus with help of the "quantum ψ -plane" \hat{q}_ψ, Q -commuting variables A, B : $[A, B]_{\hat{q}_\psi, Q} = 0$ exactly in the same way as it was done by Cigler and Kirschenhofer in [11], [12].

For that to do let us consider appropriate generalization of $A = \hat{x}$ and $B = y\hat{Q}$ where this time the action of \hat{Q} on $\{x^n\}_0^\infty$ is to be found from the condition

$$AB - \hat{q}_\psi BA \equiv [A, B]_{\hat{q}_\psi} = 0.$$

Acting with $[A, B]_{\hat{q}_\psi}$ on $\{x^n\}_0^\infty$ one easily sees that due to $\hat{q}_\psi x^n = \frac{(n+1)_\psi - 1}{n_\psi} x^n$; $n \geq 0$, $\hat{Q}x^n = b_n x^n$ where $b_0 = 0$ and $b_n = \prod_{k=1}^n \frac{(k+1)_\psi - 1}{k_\psi}$ for $n > 0$ is the solution of the difference equation: $b_n - b_{n-1} \frac{(n+1)_\psi - 1}{n_\psi} = 0$; $n > 0$.

With all above taken into account one immediately verifies that for our A and B \hat{q}_ψ -commuting variables already

$$(3.4) \quad (A + B)^n \neq \sum_{k \geq 0} \binom{n}{k}_\psi A^k B^{n-k}$$

unless $\psi_n(q) = \frac{1}{R(q^n)!}$; $R(x) = \frac{1-x}{1-q}$ hence $\hat{q}_{\psi,Q} \equiv \hat{q}_{R,\text{id}} \equiv \hat{q}_R \equiv \hat{q}_{q,\text{id}} \equiv \hat{q}_q \equiv \hat{q}$ and $\hat{q}_{\psi,Q} x^n = q^n x^n$ i.e. unless we are back to the q -case.

In conclusion one sees that the above identifications of polynomial sequence $\{p_n\}_o^\infty$ of q -binomial type and Sheffer q -polynomials $\{s_n(x)\}_{n=0}^\infty$ fail to be extended to the more general ψ -case. This means that we cannot formulate *that way* the ψ -extended finite operator calculus with help of the “quantum ψ -plane” $\hat{q}_{\psi,Q}$ -commuting variables $A, B : AB - \hat{q}_{\psi,Q}BA \equiv [A, B]_{\hat{q}_{\psi,Q}} = 0$ while considering algebra of polynomials P over the field F .

Nevertheless — already the q -case is already reach enough in abundant applications to various “ q -quantum mechanical models” — $q \equiv \omega \equiv \exp\{\frac{2\pi i}{n}\}$ case included. One may expect the natural use of q -umbral calculus in these applications to be advantageous. Models using $\hat{q}_{\psi,Q}$ -mutator $\left[Q(\partial_\psi), \hat{x}_{Q(\partial_\psi)}\right]_{\hat{q}_{R,Q}} = \text{id}$ relations are suitable play-ground for ψ -umbral calculus (leading perhaps to ψ -lasers ? — see the q -footnote in [2, p. 1887]).

For the most general cases and for further links to further readings the reader is referred to [27] and [28].

For very recent and qualitatively new applications of q -umbral and $\psi(q)$ -calculus one is referred to [29], [30], [31] and [32]. There — due to the invention of a specific $\ast\psi$ product of formal series — new families of $\psi(q)$ -extensions of Poisson processes and q -Bernoulli-Taylor formula with the rest q -term of the Cauchy type are derived among others.

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