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TRIVIAL CONSTRAINT VARIATIONAL PROBLEM

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ABSTRACT. In this paper we study trivial constraint Cartan forms arising from the variational theory of mechanical systems with non-holonomic constraints proposed recently by Krupková and Musilová. After remarks indicating obstructions in solving trivial constraint variational problem in general we discuss some special but not trivial situations.

1. INTRODUCTION

Physical systems are often subjected to various types of constraints. Clearly, it is important to derive corresponding equations of motion and then to deal with a question of their variationality. However, only the theory of holonomic constraints is satisfactorily elaborated. Non-holonomic systems are still studied by many authors with different approaches (see References).

We adopt the geometrical theory of first-order mechanical systems with non-holonomic constraints on jet manifolds developed by Krupková (see [2]) and the concept of constrained variationality proposed later by Krupková and Musilová (see [6], [7]). Using the results of these theories (briefly summarized in Section 2) we study trivial constraint Cartan forms, i.e. forms leading to identically zero left-hand sides of constrained equations of motion (Section 3). We there refer to some difficulties connected with the quite general formulation of the problem. Then we discuss some special situations.

We accept the standard notation of geometrical objects. Let $\pi : Y \to X$ be a fibred manifold, $\dim Y = m+1$, $\dim X = 1$, $\pi_r : J^rY \to X$ its r-jet prolongation (throughout this paper it will be $1 \leq r \leq 2$) and $\pi_{r,s} : J^rY \to J^sY$, $0 \leq s < r$, $J^0Y = Y$ canonical projections. Associated fibred chart on J^rY arising from the fibred chart (V, ψ) on Y, $V \subset Y$ is an open set, $\psi = (t, q^\sigma)$, $1 \leq \sigma \leq m$, takes the form (V_r, ψ_r), $V_r = \pi_{r,0}^{-1}(V)$, $\psi_1 = (t, q^\sigma, \dot{q}^\sigma)$, $\psi_2 = (t, q^\sigma, \dot{q}^\sigma, \ddot{q}^\sigma)$. A section $\delta : I \to J^rY$ of π_r, $I \subset X$ is an open set, is called *holonomic* if there is a section γ of π such that $\delta = J^r\gamma$.

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The paper is in final form and no version of it will be submitted elsewhere.

A vector field ξ on J^rY is called *π_r-projectable* if there is a vector field ξ_0 on X such that $T\pi_r\,\xi = \xi_0 \circ \pi_r$ and is called *π_r-vertical* if $T\pi_r\,\xi = 0$. A form η on J^rY is called *π_r-horizontal* if $i_\xi\eta = 0$ for every π_r-vertical vector field on J^rY and is called *contact* if $J^r\gamma^*\eta = 0$ for every section γ of π. Analogously, a $\pi_{r,s}$-projectable vector field, $\pi_{r,s}$-vertical vector field and $\pi_{r,s}$-horizontal form is defined. By means of contact 1-forms $\omega^\sigma = \mathrm{d}q^\sigma - \dot{q}^\sigma \mathrm{d}t$ and $\dot{\omega}^\sigma = \mathrm{d}\dot{q}^\sigma - \ddot{q}^\sigma \mathrm{d}t$ the basis $(\mathrm{d}t, \omega^\sigma, \mathrm{d}\dot{q}^\sigma)$ of 1-forms on J^1Y and the basis $(\mathrm{d}t, \omega^\sigma, \dot{\omega}^\sigma, \mathrm{d}\ddot{q}^\sigma)$ of 1-forms on J^2Y can be introduced. For every k-form η on J^rY it holds: $\pi^*_{r+1,r}\eta = p_{k-1}\eta + p_k\eta$ where $p_{k-1}\eta$ is $(k-1)$-*contact component* of η and $p_k\eta$ is *k-contact component* of η. A k-form η is called $(k-1)$-*contact* if $p_k\eta = 0$ and is called *k-contact* if $p_{k-1}\eta = 0$.

A *distribution* $\mathcal{D}$ on J^rY is defined as a mapping assigning to every point $x \in J^rY$ a vector subspace $\mathcal{D}(x) \subset T_xJ^rY$. A distribution can be generated either by (local) vector fields ξ_ι on J^rY, $\iota \in \mathcal{I}$, or by its *annihilators*, i.e. (local) 1-forms η_κ on J^rY, $\kappa \in \mathcal{K}$, $i_{\xi_\iota}\eta_\kappa = 0$ for every $\iota \in \mathcal{I}$, $\kappa \in \mathcal{K}$ where $\mathcal{I}$, $\mathcal{K}$ are sets of indices. We will write

$$\mathcal{D} = \mathrm{span}\,\{\,\xi_\iota \mid \iota \in \mathcal{I}\,\}\,, \quad \mathcal{D}^0 = \mathrm{span}\,\{\,\eta_\kappa \mid \kappa \in \mathcal{K}\,\}\,.$$

A section δ of π_r is called an *integral section* of a distribution $\mathcal{D}$ if $\delta^*\eta = 0$ for every $\eta \in \mathcal{D}^0$.

2. Basic geometrical concepts

This section contains a brief outline of basic geometrical concepts that we will need. For proofs and more details see [2], [3], [6], [7].

Unconstrained mechanical systems (see [2]).

We will start from the concept of *dynamical form* E, defined as a 1-contact $\pi_{2,0}$-horizontal 2-form on J^2Y. Such a form is in every fibred chart expressed as follows:

$$E = E_\sigma\,(t, q^\rho, \dot{q}^\rho, \ddot{q}^\rho)\;\mathrm{d}q^\sigma \wedge \mathrm{d}t\,, \qquad 1 \le \sigma, \rho \le m\,.$$

Having on mind physical applications we will assume that it holds

$$E_\sigma\,(t, q^\rho, \dot{q}^\rho, \ddot{q}^\rho) = A_\sigma\,(t, q^\rho, \dot{q}^\rho) + B_{\sigma\nu}\,(t, q^\rho, \dot{q}^\rho)\,\ddot{q}^\nu\,.$$

For every dynamical form E there exists a 2-form α on J^1Y, called *Lepagean form associated to* E, such that $p_1\alpha = E$, i.e.

$$\alpha = A_\sigma\,\omega^\sigma \wedge \mathrm{d}t + B_{\sigma\nu}\,\omega^\sigma \wedge \mathrm{d}\dot{q}^\nu + F_{\sigma\nu}\,\omega^\sigma \wedge \omega^\nu$$

where $F_{\sigma\nu} = -F_{\nu\sigma}$ are some functions on J^1Y. The equivalence class $[\alpha]$ with the equivalence relation $\alpha_1 \sim \alpha_2$ if $\alpha_1 - \alpha_2 = F_{\sigma\nu}\,\omega^\sigma \wedge \omega^\nu$ is called a *Lepagean class of* E, or a *first-order mechanical system associated to* E. Finally, by a *path of a dynamical form* E, or by a *path of a mechanical system* $[\alpha]$, we mean a section γ of π satisfying the condition

$$E \circ J^2\gamma = 0\,, \quad \text{i.e.} \quad A_\sigma + B_{\sigma\nu}\ddot{q}^\nu = 0 \quad \text{along} \quad J^2\gamma\,.$$

The last equations are called (*unconstrained*) *equations of motion*.

Concept of unconstrained variationality (see [2], [3]).

A dynamical form E is called (*locally*) *variational* if in a neighbourhood of every point $x \in J^1Y$ there exists a (*local*) *Lagrangian* $\lambda = L\,(t, q^\sigma, \dot{q}^\sigma)\,\mathrm{d}t$ such that

$$E_\sigma = \frac{\partial L}{\partial q^\sigma} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^\sigma}\,.$$

The equations of motion are then called *Euler-Lagrange equations.*

Theorem 1. *A dynamical form E is locally variational iff the corresponding mechanical system $[\alpha]$ contains a closed representative α_E. Such a representative is unique.*

Remark 1.

(a) The closed representative α_E in a variational mechanical system $[\alpha]$ can be locally expressed as e.g. $\alpha_E = \mathrm{d}\theta$ where a 1-form $\theta = L\,\mathrm{d}t + \frac{\partial L}{\partial \dot{q}^\sigma}\,\omega^\sigma$ is called *Cartan form.*

(b) A mechanical system $[\alpha]$ is variational iff functions A_σ, $B_{\sigma\nu}$ satisfy the well-known Helmholtz conditions.

A Lagrangian λ for which the corresponding dynamical form E vanishes identically is called a *trivial Lagrangian.*

Remark 2. Cartan forms θ_1 and θ_2 correspond to the same variational system iff $\theta_1 - \theta_2 = \mathrm{d}h$ where $h\,(t, q^\sigma)$ is a function on J^1Y.

Mechanical systems with non-holonomic constraints (see [2]).

Consider a mechanical system subjected to a system of (*local*) *non-holonomic constraints*

$$f^i\,(t, q^\sigma, \dot{q}^\sigma) = 0\,, \qquad \operatorname{rank}\left(\frac{\partial f^i}{\partial \dot{q}^\sigma}\right) = k\,, \qquad 1 \le i \le k\,, \qquad 1 \le k \le m-1\,,$$

or, in a more frequent *normal form*

$$\dot{q}^{m-k+i} - g^i\left(t, q^\sigma, \dot{q}^l\right) = 0\,, \qquad 1 \le l \le m-k\,.$$

This system determines a *constraint submanifold* Q of J^1Y fibred over Y of dimension $2m+1-k$. Obviously, a set of paths now consists only of those sections γ of π satisfying the condition $f^i \circ J^1\gamma = 0$ (so-called *Q-admissible sections*, see [5], [16]).

Let us recall concepts appearing in this theory. First, on an open set U in J^1Y, $U \cap Q \neq \emptyset$, define (*local*) *constraint 1-forms*

$$\phi^i = f^i\mathrm{d}t + \frac{\partial f^i}{\partial \dot{q}^\sigma}\,\omega^\sigma \quad \text{and} \quad \text{forms} \quad \varphi^i = \iota^*\phi^i\,, \qquad 1 \le i \le k$$

where $\iota : Q \to J^1Y$, $\iota\left(t, q^\sigma, \dot{q}^l\right) = \left(t, q^\sigma, \dot{q}^l, g^i\,(t, q^\nu, \dot{q}^s)\right)$ is canonical embedding of Q into J^1Y.

Proposition 1.

(a) *Forms $\varphi^i = \iota^*\phi^i$ where ϕ^i run over the set of all constraint 1-forms on the open sets covering Q generate a distribution $\mathcal{C}$ on Q of corank k.*

(b) *A section γ of π satisfies the condition $f^i \circ J^1\gamma = 0$ iff $J^1\gamma$ is an integral section of the distribution $\mathcal{C}$.*

Distribution $\mathcal{C}$ on Q,

$$\mathcal{C}^0 = \operatorname{span}\{\varphi^i\} = \operatorname{span}\{\iota^*\phi^i\}\,,$$

is called *canonical distribution.* Further, forms φ^i generate the *constraint ideal*

$$\mathcal{I}\left(\mathcal{C}^0\right) = \left\{\psi_i \wedge \varphi^i \mid \psi_i \text{ is a form on } Q\right\}\,.$$

Its subset of k-forms will be denoted by $\mathcal{I}_k(\mathcal{C}^0)$. Finally, taking into account the fact

$$\varphi^i = -\frac{\partial g^i}{\partial \dot{q}^l}\,\omega^l + \iota^*\omega^{m-k+i} = -\frac{\partial g^i}{\partial \dot{q}^l}\,\omega^l + \mathrm{d}q^{m-k+i} - g^i \mathrm{d}t$$

we can introduce the new basis $\left(\mathrm{d}t, \omega^l, \varphi^i, \mathrm{d}\dot{q}^l\right)$ of 1-forms on Q.

Define the equivalence relation $\iota^*\alpha_1 \sim \iota^*\alpha_2$ by $\iota^*\alpha_1 - \iota^*\alpha_2 = \bar{F}_{ls}\,\omega^l \wedge \omega^s + \chi_i \wedge \varphi^i$ where $\bar{F}_{ls} = -\bar{F}_{sl}$ are some functions on Q and χ_i is a 1-form on Q. Clearly, $\iota^*\alpha_1 \sim \iota^*\alpha_2$ iff $\alpha_1 \sim \alpha_2$, i.e. iff $\alpha_1, \alpha_2 \in [\alpha]$. The equivalence class $[\alpha_Q] = [\iota^*\alpha]$ where $\alpha_Q = \iota^*\alpha$ for some $\alpha \in [\alpha]$ is called a *constrained mechanical system on* Q *related to the mechanical system* $[\alpha]$. A direct calculation gives

$$\alpha_Q = \bar{A}_l\,\omega^l \wedge \mathrm{d}t + \bar{B}_{ls}\,\omega^l \wedge \mathrm{d}\dot{q}^s + \bar{F}_{ls}\,\omega^l \wedge \omega^s + \chi_i \wedge \varphi^i$$

where functions $\bar{A}_l$, $\bar{B}_{ls}$ depend not only on A_σ, $B_{\sigma\nu}$ but on the non-holonomic constraints g^i as well:

$$\bar{A}_l = \left[A_l + \sum_{j=1}^{k} A_{m-k+j}\frac{\partial g^j}{\partial \dot{q}^l} + \sum_{i=1}^{k}\left(B_{l,m-k+i} + \sum_{j=1}^{k} B_{m-k+j,m-k+i}\frac{\partial g^j}{\partial \dot{q}^l}\right)\left(\frac{\partial g^i}{\partial t} + \frac{\partial g^i}{\partial q^\sigma}\dot{q}^\sigma\right)\right] \circ \iota\,,$$

$$\bar{B}_{ls} = \left[B_{ls} + \sum_{i=1}^{k}\left(B_{l,m-k+i}\frac{\partial g^i}{\partial \dot{q}^s} + B_{m-k+i,s}\frac{\partial g^i}{\partial \dot{q}^l}\right) + B_{m-k+j,m-k+i}\frac{\partial g^j}{\partial \dot{q}^l}\frac{\partial g^i}{\partial \dot{q}^s}\right] \circ \iota\,.$$

A *path of a constrained system* $[\alpha_Q]$ is defined in an analogy to an unconstrained case, i.e. as a section γ of π satisfying the conditions

$$f^i \circ J^1\gamma = 0 \quad \text{and} \quad \bar{A}_l + \bar{B}_{ls}\ddot{q}^s = 0 \quad \text{along} \quad J^2\gamma\,.$$

These equations are called *reduced equations of motion.*

Remark 3. The above geometrical approach is equivalent to the physical one considering the constraint force of the (local) form

$$\Phi_{U_\iota} = \mu^i \frac{\partial f^i}{\partial \dot{q}^\sigma}\,\omega^\sigma \wedge \mathrm{d}t\,.$$

Such a force is called *Chetaev force* and functions μ^i on J^1Y are called *Lagrange multipliers.*

Concept of constrained variationality (see [6], [7]).

Concept of variationality for a system with non-holonomic constraints is defined through the property of variational unconstrained case (cf. Theorem 1).

Definition 1. A constrained mechanical system $[\alpha_Q]$ is called *variational* if it contains a closed representative.

Remark 4.

(a) A constrained system arising from the variational one is obviously variational (in the previous sense). The converse is not true in general.
(b) A constrained system is variational iff so-called *constraint Helmholtz conditions* are fulfilled (see [6], [7]).
(c) Contrary to an unconstrained case, the closed representative of a variational constrained system need not be unique. Two closed representatives differ by a closed form $\rho = \bar{F}_{ls}\, \omega^l \wedge \omega^s + \chi_i \wedge \varphi^i$.

Now, for the simplification of the following formulas put

$$\varepsilon_l = \frac{\partial_c}{\partial q^l} - \frac{\mathrm{d}'_c}{\mathrm{d}t}\frac{\partial}{\partial \dot{q}^l} - \ddot{q}^s \frac{\partial^2}{\partial \dot{q}^l \partial \dot{q}^s}, \qquad \varepsilon'_l = \frac{\partial_c}{\partial q^l} - \frac{\mathrm{d}'_c}{\mathrm{d}t}\frac{\partial}{\partial \dot{q}^l}$$

where

$$\frac{\partial_c}{\partial q^l} = \frac{\partial}{\partial q^l} + \frac{\partial g^j}{\partial \dot{q}^l}\frac{\partial}{\partial q^{m-k+j}}, \qquad \frac{\mathrm{d}'_c}{\mathrm{d}t} = \frac{\partial}{\partial t} + \dot{q}^l \frac{\partial}{\partial q^l} + g^j \frac{\partial}{\partial q^{m-k+j}}.$$

Proposition 2. *Let $[\alpha_Q]$ be a variational constrained system. Then, on Q there exists a representative $\alpha_Q \in [\alpha_Q]$ and a (local) 1-form*

$$\bar{\theta} = \bar{L}\mathrm{d}t + \frac{\partial \bar{L}}{\partial \dot{q}^l}\,\omega^l + \bar{L}_{m-k+i}\,\varphi^i \quad \text{such that} \quad \alpha_Q = \mathrm{d}\bar{\theta}$$

where $\bar{L}$ and $\bar{L}_{m-k+i}$, $1 \le i \le k$, are functions on Q.
By means of coefficients of the form $\bar{\theta}$ the reduced equations of motion can be expressed as follows:

$$f^i \circ J^1\gamma = 0, \qquad \varepsilon_l\left(\bar{L}\right) - \bar{L}_{m-k+i}\,\varepsilon_l\left(g^i\right) = 0.$$

Definition 2.

(a) A form $\bar{\theta} = \bar{L}\mathrm{d}t + \frac{\partial \bar{L}}{\partial \dot{q}^l}\,\omega^l + \bar{L}_{m-k+i}\,\varphi^i$ is called *constraint Cartan form* (cf. [4], [5], [7], [16]).
(b) A form $\bar{\tau} = \bar{T}\mathrm{d}t + \frac{\partial \bar{T}}{\partial \dot{q}^l}\,\omega^l + \bar{T}_{m-k+i}\,\varphi^i$ where $\bar{T}$ and $\bar{T}_{m-k+i}$ are functions on Q satisfying $\varepsilon_l\left(\bar{T}\right) - \bar{T}_{m-k+i}\,\varepsilon_l\left(g^i\right) \equiv 0$ is called *trivial constraint Cartan form.*

Remark 5.

(a) Note that constraint Cartan forms are given by $(k+1)$ functions $\bar{L}$, $\bar{L}_{m-k+i}$ on Q.
(b) Constraint Cartan forms $\bar{\theta}_1$ and $\bar{\theta}_2$ correspond to the same variational system iff $\bar{\theta}_1 - \bar{\theta}_2 = \bar{\tau}$.

3. Trivial constraint Cartan forms

After a brief summary of basic geometrical concepts we will discuss trivial constraint variational problem. First, we will demonstrate that its quite general formulation is not elementary. Then we will discuss some special situations.

General remarks.

Proposition 3. *All trivial constraint Cartan forms are given by closed forms* $\rho = \bar{F}_{ls}\,\omega^l \wedge \omega^s + \chi_i \wedge \varphi^i$ *as* $\mathrm{d}\bar{\tau} = \rho$.

Obviously, there always exist trivial constraint Cartan forms corresponding to $\rho = 0$, i.e. $\bar{\tau} = \mathrm{d}h$ where $h\,(t, q^\sigma)$ is a function on Q. On the other hand, for concrete choice of constraints we can ask if there exist closed forms $\rho \neq 0$ as well. (Trivial constraint variational problem is closely connected with a question of uniqueness of the closed representative in a variational constrained mechanical system.)

Necessary and sufficient conditions for the existence of a closed form $\rho \neq 0$ are given by constraint Helmholtz conditions (see [6], [7]) in which $\bar{A}_l = 0$, $\bar{B}_{ls} = 0$ for every $1 \leq l,\ s \leq m-k$. These conditions represent a system of PDE for coefficients in the form ρ. Write down their quite general solution (i.e. write down form ρ explicitly) is, of course, impossible. For this reason we will focus on some concrete situations only and we will study a question of existence of a closed form $\rho \neq 0$. First, let us summarize some useful formulas.

From the definition of trivial constraint Cartan form $\varepsilon_l\left(\bar{T}\right) - \bar{T}_{m-k+i}\ \varepsilon_l\left(g^i\right) \equiv 0$ where $\bar{T}$, $\bar{T}_{m-k+i}$ are functions on Q we have

$$\varepsilon_l'\left(\bar{T}\right) = \bar{T}_{m-k+i}\,\varepsilon_l'\left(g^i\right)\ , \tag{1}$$

$$\frac{\partial^2 \bar{T}}{\partial \dot{q}^s \partial \dot{q}^l} = \bar{T}_{m-k+i}\,\frac{\partial^2 g^i}{\partial \dot{q}^s \partial \dot{q}^l} \tag{2}$$

and

$$\begin{aligned}
\mathrm{d}\bar{\tau} = \rho = \frac{1}{2}\Big[\frac{\partial \bar{T}_{m-k+i}}{\partial \dot{q}^s}\,\varepsilon_l'\left(g^i\right)\Big]_{\mathrm{alt}(l,s)}\omega^l \wedge \omega^s + \Big[\frac{\partial \bar{T}_{m-k+i}}{\partial \dot{q}^l}\Big]\mathrm{d}\dot{q}^l \wedge \varphi^i \\
+ \Big[\frac{\mathrm{d}_c'\bar{T}_{m-k+i}}{\mathrm{d}t} + \bar{T}_{m-k+j}\,\frac{\partial g^j}{\partial q^{m-k+i}} - \frac{\partial \bar{T}}{\partial q^{m-k+i}}\Big]\mathrm{d}t \wedge \varphi^i \\
+ \Big[\frac{\partial_c \bar{T}_{m-k+i}}{\partial q^l} + \bar{T}_{m-k+j}\,\frac{\partial^2 g^j}{\partial \dot{q}^l \partial q^{m-k+i}} - \frac{\partial^2 \bar{T}}{\partial \dot{q}^l \partial q^{m-k+i}}\Big]\omega^l \wedge \varphi^i \\
+ \Big[\frac{\partial \bar{T}_{m-k+i}}{\partial q^{m-k+j}}\Big]_{\mathrm{alt}(i,j)}\varphi^j \wedge \varphi^i
\end{aligned} \tag{3}$$

where the first term of (3) was modified using the partial derivative of (1) with respect to $\dot{q}^s$.

For a given point of Q, (1) and (2) can be regarded as a homogeneous system of $N = (m-k)\left(1 + \frac{m-k+1}{2}\right)$ linear equations for $k + N$ unknown values $\bar{T}_{m-k+i}$, $\frac{\partial^2 \bar{T}}{\partial \dot{q}^s \partial \dot{q}^l}$, $\varepsilon_l'\left(\bar{T}\right)$. This system has the rank N and thus its solutions generate a k-dimensional vector space. For example, functions $\frac{\partial^2 \bar{T}}{\partial \dot{q}^s \partial \dot{q}^l}$ and $\varepsilon_l'\left(\bar{T}\right)$ could be expressed as linear combinations of free functions $\bar{T}_{m-k+1}, \ldots, \bar{T}_m$. Clearly, there arises a problem to solve a system of partial differential equations with respect to $\bar{T}$. The solution can be obtained in quite special situations only, not in general.

On the other hand, it can be seen from (3) that there are special types of constraints such that $\rho \in \mathcal{I}_2(\mathcal{C}^0)$ for arbitrary $\bar{T}_{m-k+i}$. Such constraints are given by condition

$\varepsilon'_l(g^i) = 0$ for every $1 \leq i \leq k$, $1 \leq l \leq m-k$. This condition is fulfilled especially for semiholonomic constraints and for constraints depending only on velocities. Let us discuss these situations more precisely.

Semiholonomic constraints.

A system of non-holonomic constraints is called *semiholonomic constraints* if the constraint ideal $\mathcal{I}(\mathcal{C}^0)$ is differential, i.e. if $\mathrm{d}\varphi^i \in \mathcal{I}_2(\mathcal{C}^0)$ for every $1 \leq i \leq k$. From the definition following necessary and sufficient conditions for the constraints to be semiholonomic can be obtained:

$$\varepsilon'_l\left(g^i\right) = 0\,, \qquad \frac{\partial^2 g^i}{\partial \dot{q}^s \partial \dot{q}^l} = 0\,, \qquad 1 \leq i \leq k\,, \qquad 1 \leq l, s \leq m-k\,.$$

Then, from conditions (1) and (2) we have

$$\bar{T} = a_l\, \dot{q}^l + b$$

where a_l and b are functions of (t, q^σ) which fulfil $\varepsilon'_l\left(\bar{T}\right) = 0$. Functions $\bar{T}_{m-k+i}$ can be chosen arbitrarily, i.e. $\rho \neq 0$ in general. Two special possibilities for $\bar{\tau}$ can be considered:

$$\bar{\tau} = \bar{T}\mathrm{d}t + \frac{\partial \bar{T}}{\partial \dot{q}^l}\,\omega^l = \left(a_l\, \dot{q}^l + b\right)\mathrm{d}t + a_l\,\omega^l\,, \qquad \bar{\tau} = \bar{T}_{m-k+i}\,\varphi^i\,.$$

Constraints dependent on velocities.

Consider $g^i = g^i\left(\dot{q}^l\right)$ for every $1 \leq i \leq k$. Then again $\varepsilon'_l\left(g^i\right) = 0$ for $1 \leq i \leq k$, $1 \leq l \leq m-k$. Thus, $\rho \in \mathcal{I}_2(\mathcal{C}^0)$,

$$\rho = \left[\frac{\partial \bar{T}_{m-k+i}}{\partial \dot{q}^l}\right]\mathrm{d}\dot{q}^l \wedge \varphi^i + \left[\frac{\mathrm{d}'_c \bar{T}_{m-k+i}}{\mathrm{d}t} - \frac{\partial \bar{T}}{\partial q^{m-k+i}}\right]\mathrm{d}t \wedge \varphi^i$$
$$+ \left[\frac{\partial_c \bar{T}_{m-k+i}}{\partial q^l} - \frac{\partial^2 \bar{T}}{\partial \dot{q}^l \partial q^{m-k+i}}\right]\omega^l \wedge \varphi^i + \left[\frac{\partial \bar{T}_{m-k+i}}{\partial q^{m-k+j}}\right]_{\mathrm{alt}(i,j)} \varphi^j \wedge \varphi^i$$

where $\bar{T}$, $\bar{T}_{m-k+i}$ are solutions of (1), (2).

In special cases of the constraints it may happen that for all solutions of (1), (2) it holds $\rho = 0$. Then, all trivial constraint Cartan forms are given by exterior derivatives of the functions on Q, i.e. $\bar{\tau} = \mathrm{d}h$. In such a case the closed representative of *every* variational constrained mechanical system $[\alpha_Q]$ is unique. Let us present concrete example in which the conclusion $\rho = 0$ can be obtained e.g. most simply from constraint Helmholtz conditions:

$$m = 3\,, \quad k = 1\,, \quad g^1 = \dot{q}^3 = \mathrm{const} \cdot \left[\left(\dot{q}^1\right)^2 + \left(\dot{q}^2\right)^2\right].$$

Note that for $m = 2$ and analogous constraint $g^1 = \dot{q}^2 = \mathrm{const} \cdot \left(\dot{q}^1\right)^2$ a closed form $\rho \neq 0$ exists, e.g. $\rho = \mathrm{d}\dot{q}^1 \wedge \varphi^1$.

The same conclusion $\rho = 0$ we obtain also e.g. for

$$m = 3\,, \quad k = 1\,, \quad g^1 = \dot{q}^3 = \mathrm{const} \cdot \left(\dot{q}^1 \dot{q}^2\right).$$

On the other hand, for

$$m = 3\,, \quad k = 1\,, \quad g^1 = \dot{q}^3 = \mathrm{const} \cdot \left(\dot{q}^1 + \dot{q}^2\right)^2$$

a closed form $\rho \neq 0$ exists, e.g. $\rho = \mathrm{d}\dot{q}^1 \wedge \varphi^1 + \mathrm{d}\dot{q}^2 \wedge \varphi^1$.

4. Conclusion

The main purpose of this paper was to study trivial constraint variational problem. As we have shown in Section 3, quite general solution of this problem appears not to be elementary. Thus, every concrete choice of non-holonomic constraints has to be studied separately with appropriately chosen approach.

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