Edward Grzegorek

Always of the first category sets (II)

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Results of this note were presented during 13th Winter School on Abstract Analysis in Czechoslovakia. We investigated in [5] and [6] a useful sub-σ-ideal, denoted by $\overline{\mathcal{K}^*}$, of the σ-ideal of subsets of the real line R which are always of the first category, denoted by $\mathcal{K}^*$. Now we prove that each λ-set in the sense of [8] belongs to $\overline{\mathcal{K}^*}$. We also obtain as a corollary of a result of [6] elimination of the assumption CH in the theorem of Sierpiński [16] that there is a continuous $f: R \to R$ such that there exists $A \in \mathcal{K}^*$ for which $f(A)$ does not have Baire property in the restricted sense (it also shows that Proposition C_{46} in [14] is simply a theorem of ZFC). We also strengthen the theorem of Sierpiński [15] that there is an uncountable subset X of R such that all its Borel isomorphic images into R are in $\mathcal{K}^*$ and have Lebesgue measure zero. Moreover we remove a mistake in our proof of Theorem 1 in [6].

Let X be a separable metric space. If every dense in itself subset of X is of the first category relative to itself, then X is said to be always of the first category. We denote by $\mathcal{K}(X)$ or simply $\mathcal{K}$ if X=R, the σ-ideal of subsets of X which are of the first category in X and by $\mathcal{K}^*(X)$, or $\mathcal{K}^*$ if X=R, the σ-ideal of subsets of X which are always of the first category. A subset A of X has the Baire property ($A \in \mathcal{B}_w(X)$) if there exists an open subset Q of X such that $A \setminus Q \in \mathcal{K}(X)$ and $Q \setminus X \in \mathcal{K}(X)$. A subset A of X has the Baire property in the restricted sense ($A \in \mathcal{B}_r(X)$) if for every subset B of X we have $B \cap A \in \mathcal{B}_w(B)$. If X is a separable complete metric space then for every $A \subseteq X$ we have $A \in \mathcal{K}^*(X)$ iff $\mathcal{P}(A) \subseteq \mathcal{B}_r(X)$ [8]. We denote by λ the family of subsets X of R such that every countable subset of X is a $\mathcal{G}_\delta$ set in X [8]. We denote by $\mathcal{B}(X)$ the σ-field of Borel subsets of X. A space X is called a universal null set if there is no continuous probability measure on $\mathcal{B}(X)$. We denote by $\mathcal{L}_0$ the

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This paper is in final form and no version of it will be submited for publication elswhere.

σ-ideal of Lebesgue measure zero subsets of R. There are survey articles [2] and [10] concerning the above notions. A family $\mathcal{J}$ of subsets of the real line R is called σ-ideal on R if $A_0, A_1, A_2, \ldots \in \mathcal{J}$ implies $\bigcup\{A_n: n=0,1,2,\ldots\} \in \mathcal{J}$ and $\mathcal{P}(A_0) \subseteq \mathcal{J}$, $\mathcal{J} \subsetneq \mathcal{P}(R)$ and for every $x \in R$ we have $\{x\} \in \mathcal{J}$. If $\mathcal{J}$ is a σ-ideal on R then we define (see [6])

$$\overline{\mathcal{J}} = \{A \subseteq R: \text{ for every } B \subseteq R \text{ such that there exists a 1-1 Borel measurable function } f: B \to A \text{ we have } B \in \mathcal{J}\}.$$

It is clear that $\overline{\mathcal{J}}$ is a σ-ideal on R such that $\overline{\mathcal{J}} \subseteq \mathcal{J}$. We will need the following theorem concerning $\overline{\mathcal{K}^*}$.

<u>Theorem 1</u> ([6]). Let $m_1 = \min\{|Y| : Y \subseteq R \text{ and } Y \notin \mathcal{K}\}$. There is $X \subseteq R$ such that $|X| = m_1$ and $X \in \overline{\mathcal{K}^*}$.

<u>Remark.</u> We would like to remove a mistake in our proof of Theorem 1 in [6]. A reader who is interested in the proof of Theorem 1 in [6] should replace lines 18-24 on page 142 in [6] by the following " Let $F_\alpha = \bigcup\{F_n^\alpha : n<\omega\}$ where F_n^α are closed in Y. Setting

$$E_i^n = \{\alpha<m_1 : O_i \subseteq Y \setminus F_n^\alpha\} \text{ for every } i<\omega \text{ and every } n<\omega$$

we get

$$Z = (m_1 \times Y) \setminus \bigcap_{n<\omega} \Big(\bigcup_{i<\omega} E_i^n \times O_i\Big) \quad (\text{compare } [1]).$$

Let $\mathcal{A}$ be a countably generated and separating points σ-field on m_1. Let $\mathcal{C}$ be a σ-field on m_1 generated by $\mathcal{A}$ and the family $\{E_i^n : i,n<\omega\}$. It is clear that Z belongs to the product σ-field"

Sierpiński proved (see [16]), assuming CH, that there exists a continuous function $f: R \to R$ such that there exists $X \in \mathcal{K}^*$ with $f(X) \notin \mathcal{B}_r$ (and such that the restriction of f to X is 1-1). This theorem is true in ZFC. Namely we have the following

<u>Theorem 2.</u> There is $X \in \overline{\mathcal{K}^*}$ such that there is a continuous function $f: R \to R$ with $f(X) \notin \mathcal{B}_w$. We can additionally have that f restricted to X is 1-1.

Indeed, since for every $A \subseteq R$ we have $A \in \mathcal{K}$ iff $\mathcal{P}(A) \subseteq \mathcal{B}_w$ [8] it easily follows from Proposition 4 in [6] that there is $Y \in \overline{\mathcal{K}^*}$ and there is a continuous 1-1 function $f: Y \to R$ with $f(Y) \notin \mathcal{B}_w$.

Now Theorem 2 follows from the following theorem of Sierpiński (Corollary 2 in [17]).

Let $\mathcal{F}$ be a family of subsets of R such that for every $F \in \mathcal{F}$ we have:

$g(F) \in \mathcal{F}$ for every homeomorphism g from F into R,

$(F \cup A) \setminus B \in \mathcal{F}$ for every countable $A, B \subseteq R$.

Then

$\{g(F): F \in \mathcal{F}$ and $g: F \to R$ is a 1-1 continuous function$\}$ =
$\{g(F): F \in \mathcal{F}$ and $g: R \to R$ is a continuous function such that f restricted to F is 1-1$\}$.

A similar theorem for universal null sets can be found in [4]. Add that Theorem 2 also shows that Proposition C_{46} in [14] is simply a theorem of ZFC.

It is clear that $\overline{\mathcal{K}^*} \subseteq \mathcal{K}^*$ and it is known (compare Remark1 in [6]) that assuming CH (or Martin's Axiom) $\overline{\mathcal{K}^*} \subsetneq \mathcal{K}^*$. We have the following

<u>Theorem 3.</u> $\lambda \subsetneq \overline{\mathcal{K}^*}$.

<u>Proof.</u> We need the following

<u>Lemma 1.</u> Let $(\mathcal{K})_c = \{A \subseteq R:$ for every $B \subseteq R$ such that there exists a 1-1 continuous function $f: B \to A$ we have $B \in \mathcal{K}\}$. Then $(\mathcal{K})_c = \overline{\mathcal{K}^*}$.

<u>Proof.</u> Since $\overline{\mathcal{K}^*} = \overline{\mathcal{K}}$ (see Proposition 3 in [6]) in order to prove Lemma 1 it is enough to prove $(\mathcal{K})_c = \overline{\mathcal{K}}$. It is clear that $\overline{\mathcal{K}} \subseteq (\mathcal{K})_c$. Let $A \in (\mathcal{K})_c$. In order to prove $A \in \overline{\mathcal{K}}$ consider $B \subseteq R$ such that there is a 1-1 Borel measurable function $f: B \to A$. There are B_1, B_2 such that $B = B_1 \cup B_2$, $B_1 \in \mathcal{K}(B)$ and the restriction g of f to B_2 is continuous [8]. We have $g: B_2 \to A$ is a 1-1 continuous function and $A \in (\mathcal{K})_c$. Hence $B_2 \in \mathcal{K}$ and $B \in \mathcal{K}$, so $A \in \overline{\mathcal{K}}$.

<u>Lemma 2</u> (see [8] or [10]).

a) $\lambda \subseteq \mathcal{K}^*$.

b) Let $X, Y \subseteq R$ be such that there is a 1-1 continuous function on X into Y. Then if $Y \in \lambda$ then $X \in \lambda$.

Now let $X \in \lambda$. By Lemma 2, $X \in (\mathcal{K})_c$. Hence by Lemma 1, $X \in \overline{\mathcal{K}^*}$. We have proved $\lambda \subseteq \overline{\mathcal{K}^*}$. The fact that $\lambda \subsetneq \overline{\mathcal{K}^*}$ follows e.g. from $\lambda \subseteq \overline{\mathcal{K}^*}$

and the fact that $\widetilde{\mathcal{K}^*}$ is a σ-ideal on R whereas λ is known not to be even finite additive (Rothberger [12], compare [8] and [10]).

We strengthen the following

<u>Theorem</u> (Sierpiński, Theorem 5 in [15]). There exists uncountable subset $A \subseteq R$ such that each set $B \subseteq R$ which is Borel isomorphic with A satisfies $B \in \mathcal{L}_0 \cap \mathcal{K}^*$.

Recall that Sierpiński proved that each selector from nonempty constituents of a coanalytic non-Borel set has the property as in the above Theorem. Hence A in the proof of Sierpiński necessary has cardinality $\aleph_1$. We have the following (compare Theorem 3 in [5]).

<u>Theorem 4.</u> Let $m_1 = \min\{|X|: X \notin \mathcal{K}\}$, let $m_2 = \min\{|X|: X \notin \mathcal{L}_0\}$ and let $m = \min\{m_1, m_2\}$. There is $A \subseteq R$ with $|A| = m$ and for every Borel isomorphism $f: A \to R$ we have $f(A) \in \mathcal{L}_0 \cap \mathcal{K}^*$. Moreover instead of that f is Borel isomorphism we can assume that $f^{-1}: f(A) \to A$ is Borel measurable (and f is 1-1).

Instead of Theorem 4 we prove more general

<u>Theorem 4*.</u> Let $\{\mathcal{J}_t: t \in T\}$ be a family of σ-ideals on R and let n be such that for every $t \in T$ there is $A_t \in \widetilde{\mathcal{J}_t}$ with $|A_t| = n$. Then there is $A \in \bigcap\{\widetilde{\mathcal{J}_t}: t \in T\}$ such that:

a) if $|T| \leq \aleph_0$, then we can have $|A| = n$,

b) if $|T| \leq \aleph_1$, then we can have $|A| = \min\{\aleph_1, n\}$,

c) if Martin's Axiom holds and $|T| \leq 2^{\aleph_0}$, then we can have $|A| = n$.

<u>Proof.</u> a) Choose for every $t \in T$ an $A_t \in \widetilde{\mathcal{J}_t}$ such that $|A_t| = n$. Let X be an abstract set such that $|X| = n$ and let for every $t \in T$ $f_t: A_t \to X$ be a 1-1 onto function. Let $\mathcal{A}$ be a countably generated σ-field on X containing $f_t(\mathcal{B}(A_t))$ for every $t \in T$. In case a) we can take simply $\mathcal{A}$ = the σ-field generated by the family $\bigcup\{f_t(\mathcal{B}(A_t)): t \in T\}$. Let $g: X \to R$ be a characteristic function of a countable sequence of sets generating $\mathcal{A}$ [18]. Define $A = g(X)$. We claim that $A \in \bigcap\{\widetilde{\mathcal{J}_t}: t \in T\}$. Let $t \in T$ and let $B \subseteq R$ be such that there is a Borel measurable 1-1 function $f: B \to A$. Observe that $(f_t^{-1} g^{-1} f): B \to A_t$ is a 1-1 Borel measurable function. Hence we have $B \in \mathcal{J}_t$ because $A_t \in \widetilde{\mathcal{J}_t}$.

b) Choose for every $t \in T$ an $A_t \in \widetilde{\mathcal{J}_t}$ such that $A_t = \min\{n, \aleph_1\}$.

Let f_t and X be such as in the case a). Choose for each $t \in T$ a countable family $\mathcal{C}_t$ generating the σ-field $f_t(\mathcal{B}(A_t))$. We have $|\bigcup\{\mathcal{C}_t: t \in T\}| \leq \aleph_1$. Hence by a theorem of Rao [11] there exists a countably generated σ-field $\mathcal{A}$ on X such that $\mathcal{C}_t \subseteq \mathcal{A}$ for every $t \in T$. Hence $f(\mathcal{B}(A_t)) \subseteq \mathcal{A}$ for every $t \in T$. The rest of the proof is as in case a).

c) The proof is similar to a) and b) but to have a countably generated σ-field $\mathcal{A}$ we use the following facts. It is known [9] that if Martin's Axiom holds and $|X| < 2^{\aleph_0}$ then $\mathcal{P}(X)$ is a countably generated σ-field on X. Rao [11] and Bing, Bledsoe and Mauldin [1] proved that for every set X such that $\mathcal{P}(X) \otimes \mathcal{P}(X) = \mathcal{P}(X \times X)$ we have that if $\mathcal{F} \subseteq \mathcal{P}(X)$ and $|\mathcal{F}| \leq |X|$ then there is a countably generated σ-field $\mathcal{A}$ on X with $\mathcal{F} \subseteq \mathcal{A}$. Kunen (see [7] or [13]) proved that if we assume Martin's Axiom then $\mathcal{P}(X) \otimes \mathcal{P}(X) = \mathcal{P}(X \times X)$ for every X with $|X| \leq 2^{\aleph_0}$. (For X such that $|X| \leq \aleph_1$ the last statement is a theorem of ZFC, [11] or [7].)

Theorem 4 follows from Theorem 4^* a) because it is known that there is $A_1 \in \mathcal{K}^*$ such that $|A_1| = m_1$ [6] and there is $A_2 \in \overline{\mathcal{L}_0}$ such that $|A_2| = m_2$ ([3], compare [6]).

Remark. If $X \subseteq R$ and all Borel isomorphic images of X into R are in $\mathcal{L}_0 \cap \mathcal{K}$ then all Borel isomorphic images of X have to be in $\mathcal{N} \cap \mathcal{K}^*$, where $\mathcal{N}$ denotes the σ-ideal of universal null subsets of R [6]. Recall that it is well known that $\overline{\mathcal{L}_0} = \mathcal{N}$ (compare e.g. [2]).

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INSTITUTE OF MATHEMATICS, GDAŃSK UNIVERSITY
(INSTYTUT MATEMATYKI UNIVERSYTETU GDANSKIEGO)
ul. WITA STWOSZA 57, PL-80-952, Gdańsk, POLAND