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ON COMPLETENESS OF THE REPRESENTATION SPACE OF J. W. CALKIN

K.-D. Kürsten

Abstract: The representation space of J. W. Calkin [1] depends on a state ω on l_∞ with $\omega(c_0) = \{0\}$. We prove that this space is complete if and only if ω is a finite convex combination of ultrafilter-limits.

For a Hilbert space H (supposed infinite-dimensional, but not necessarily separable), let $B(H)$ denote the algebra of all bounded linear mappings of H into itself. Let $K(H)$ denote the ideal of all compact operators in $B(H)$. J. W. Calkin [1] has constructed a representation of the factor algebra $B(H)/K(H)$ in a certain scalar product space \mathcal{L}' . The space \mathcal{L}' depends on a state ω on l_∞ . It is known that \mathcal{L}' is complete if ω is a limit with respect to a free ultrafilter [3]. Thus, the assertion that \mathcal{L}' is never complete ([1], theorem 4.1.) is not true. In this note we obtain a necessary and sufficient condition for the completeness of \mathcal{L}' . In particular, \mathcal{L}' is not always complete.

We now fix some notations and recall the definition of \mathcal{L}' . Let ω be a positive linear functional on l_∞ such that $\omega(c_0) = \{0\}$ and $\omega((1,1,1,\dots)) = 1$. By [2] II §2 (8), there is a positive finitely additive measure μ on the σ -algebra of all subsets of \mathbb{N} such that

$$\omega((x_n)) = \int_{\mathbb{N}} x_n d\mu(n)$$

for all sequences (x_n) in l_∞ . Denote by \mathcal{L}'' the space of all H -valued sequences (f_n) tending weakly to zero. The linear structure of \mathcal{L}'' is defined by

$$a(f_n) + b(g_n) = (af_n + bg_n).$$

We consider the linear subspace

$$\mathcal{N}_\omega := \{(f_n) \in \mathcal{L}'': \omega(((f_n, f_n))) = 0\}$$

and the factor space

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$$\mathcal{L}'_\omega := \mathcal{L}''/\mathcal{N}_\omega.$$

The image of (f_n) under the quotient map $\mathcal{L}'' \rightarrow \mathcal{L}'_\omega$ is denoted by $(f_n)_\omega$. For $(f_n), (g_n) \in \mathcal{L}''$, we define

- (1) $((f_n)_\omega, (g_n)_\omega) := \omega(((f_n, g_n)))$,
- (2) $\|(f_n)_\omega\| := ((f_n)_\omega, (f_n)_\omega)^{1/2}$,
- (3) $\| (f_n) \| := \sup \{ \|f_n\| : n \in \mathbb{N} \}$,
- (4) $\| (f_n)_\omega \| := \inf \{ \| (f_n) - (h_n) \| : (h_n) \in \mathcal{N}_\omega \}$.

It is easy to see that $\mathcal{L}'_\omega, \mathcal{N}_\omega$ are closed subspaces of the space of all bounded H -valued sequences with respect to the norm (3). Consequently, the spaces $(\mathcal{L}'_\omega, \| \cdot \|)$ and $(\mathcal{L}'_\omega, \| \cdot \|)$ are Banach spaces. If $(g_n) \in \mathcal{N}_\omega$, then

$$\|(f_n)_\omega\|^2 = \|(f_n - g_n)_\omega\|^2 = \omega(\|(f_n - g_n)\|^2) \leq \| (f_n - g_n) \|^2$$

for all $(f_n) \in \mathcal{L}''$. This implies

$$\|F\| \leq \|F\|$$

for all $F \in \mathcal{L}'_\omega$. Now the open mapping theorem yields the following criterion.

Lemma: The normed space $(\mathcal{L}'_\omega, \| \cdot \|)$ is complete if and only if there exists a positive δ such that

$$(5) \quad \delta \|F\| \leq \|F\|$$

for all $F \in \mathcal{L}'_\omega$.

We now are in a position to prove the necessary and sufficient condition for the completeness of \mathcal{L}'_ω .

Theorem: The space \mathcal{L}'_ω endowed with the scalar product (1) is a Hilbert space if and only if ω is a finite convex combination of limits with respect to free ultrafilters.

Proof: Suppose $(\mathcal{L}'_\omega, (.,.))$ is a Hilbert space, i. e., $(\mathcal{L}'_\omega, \| \cdot \|)$ is complete. Let (e_n) be an orthonormal sequence in H . Denote by $\chi_n(M)$ the characteristic function of $M \subset \mathbb{N}$. Then $(\chi_n(M)e_n) \in \mathcal{L}''$ for all subsets $M \subset \mathbb{N}$. If $\mu(M) > 0$ and $(h_n) \in \mathcal{N}_\omega$, then

$$\inf \{ \|h_n\|^2 : n \in M \} \leq (\mu(M))^{-1} \omega(\|\chi_n(M)e_n\|^2) = 0.$$

This implies $\|(\chi_n(M)e_n)_\omega\| = 1$. On the other hand,

$$\|(\chi_n(M)e_n)_\omega\|^2 = \omega(\|\chi_n(M)\|) = \mu(M).$$

According to the lemma, this implies

$$0 < \inf \{ \mu(M) : M \subset \mathbb{N} \text{ and } \mu(M) > 0 \}.$$

Standard arguments yield the existence of pairwise disjoint subsets

M_1, M_2, \dots, M_l of \mathbb{M} satisfying the following conditions:

$$a_k := \mu(M_k) > 0, \quad \sum_{k=1}^l a_k = 1, \quad \text{and } M \subset M_k \text{ implies } \mu(M) \in \{0, a_k\}.$$

For $1 \leq k \leq l$, we define ultrafilters \mathcal{U}_k by

$$\mathcal{U}_k = \{M \subset \mathbb{M} : \mu(M \cap M_k) = a_k\}.$$

Since $\omega(a_0) = \{0\}$, $\mu(M) = 0$ if M is finite. Consequently the ultrafilters \mathcal{U}_k are free. For each subset $M \subset \mathbb{M}$ we have

$$\begin{aligned} \mu(M \setminus \bigcup M_k) &\leq \mu(\mathbb{M} \setminus \bigcup M_k) = 1 - \sum a_k = 0, \\ \omega((\chi_n(M))) &= \mu(M) = \mu(M \setminus \bigcup M_k) + \sum \mu(M \cap M_k) = \\ &= \sum a_k \lim_{\mathcal{U}_k} \chi_n(M). \end{aligned}$$

This implies

$$(6) \quad \omega((x_n)) = \sum_{k=1}^l a_k \lim_{\mathcal{U}_k} x_n$$

for all $(x_n) \in l_\infty$.

Conversely, suppose ω has the representation (6), whereat $a_k > 0$, $\sum a_k = 1$, and \mathcal{U}_k are free ultrafilters. Put $\delta = (\min\{a_k : 1 \leq k \leq l\})^{1/2}$. By the lemma, it suffices to verify (5) for all $F = (f_n)_\omega \in \mathcal{L}'_\omega$. For, fix $(f_n) \in \mathcal{L}''$ and $\varepsilon > 0$. Denote

$$b_k = \lim_{\mathcal{U}_k} \|f_n\|, \quad b = \max\{b_k : 1 \leq k \leq l\}.$$

By the definition of the limit with respect to an ultrafilter, there are sets $M_k \in \mathcal{U}_k$ such that $|\|f_n\| - b_k| < \varepsilon$ for all $n \in M_k$. Define $g_n := \chi_n(M_0) f_n$, where $M_0 := \bigcup M_k$. Then $(g_n - f_n) \in \mathcal{N}_\omega$ and $\|(g_n)\| \leq \varepsilon + b$. This implies

$$\delta \|(f_n)_\omega\| \leq \delta(\varepsilon + b) \leq \delta \varepsilon + (\sum a_k b_k^2)^{1/2} = \delta \varepsilon + \|(f_n)_\omega\|.$$

Letting $\varepsilon \rightarrow 0$, we get (5), which completes the proof.

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