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In: Zdeněk Frolík and Vladimír Souček and Jiří Vinárek (eds.): Proceedings of the 13th Winter School on Abstract Analysis, Section of Topology. Circolo Matematico di Palermo, Palermo, 1985. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 11. pp. [77]--79.

Persistent URL: <http://dml.cz/dmlcz/701881>

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A REMARK ON MONOIDAL CLOSED STRUCTURES ON TOP*

M.C. Pedicchio and F. Rossi

Introduction. The aim of this note is to characterize monoidal closed structures (in the sense of [2]) on the category Top of topological spaces and continuous maps.

We prove that any of such structures *must* satisfy the following conditions:

- the unit object is the singleton space;
- the tensor product has, as underlying set, the product set;
- the internal hom has, as underlying set, the set of continuous functions.

We heard that H. Niederle (see J. Činčura in [1]), in an unpublished paper, found similar results for some concrete categories, but, it seems that *symmetry is an essential condition* of his hypotheses.

Notation. By $U: \text{Top} \rightarrow \text{Set}$ we denote the canonical forgetful functor.

For any $A, B \in \text{Top}$:

- $\tau(A, B)$ stands for the set of continuous maps from A to B;
- $A \times B$ stands for the topological product;
- $A \otimes B$ stands for the product set $UA \times UB$, provided with the topology of separate continuity;
- $\langle A, B \rangle$ stands for the set $\tau(A, B)$ provided with the topology of pointwise convergence.

Proposition. If $(\square, I, \alpha, \lambda, \rho, [-, -])$ is a monoidal closed structure (not necessarily symmetric) on Top, then

- a) $UI = \{*\}$;
- b) the underlying set of $[A, B]$ is, up to natural isomorphisms $\tau(A, B)$;

* "This paper is in final form and no version of it will be submitted for publication elsewhere".

- c) the underlying set of $A \square B$ is, up to natural isomorphisms $UA \times UB$;
- d) $A \times B \leq A \square B$;
- e) the isomorphisms α, λ, ρ have canonical underlying functions;
- f) $A \square B \leq A \otimes B$;
- g) if $\pi: \tau(A \square B, C) \cong \tau(A, [B, C])$ is the adjunction of the closed structure, then, for any $f \in \tau(A \square B, C)$, πf is defined by

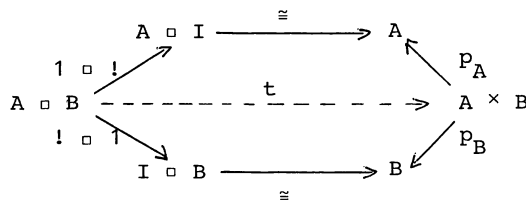
$$\pi f(a)(b) = f(a, b) \quad , \quad a \in A, b \in B;$$
 and, for any $g \in \tau(A, [B, C])$, $\pi^{-1}g$ is defined by

$$\pi^{-1}g(a, b) = g(a)(b) \quad , \quad a \in A, b \in B;$$
- h) $\langle A, B \rangle \leq [A, B]$.

Proof. a) Since $\tau(I, I)$ is a commutative monoid, then $\text{card } UI \leq 1$. It is easy to see that it cannot be zero, and the result follows.
 b) It suffices to recall that U is a representable functor, with representing object I .
 c) Let $s: UA \times UB \rightarrow U(A \square B)$ be the function defined by:

$$\begin{array}{ccc}
 UA \times UB & \xrightarrow{\quad s \quad} & U(A \square B) \\
 \cong & & \cong \\
 \tau(I, A) \times \tau(I, B) & \xrightarrow{\quad \bar{s} \quad} & \tau(I, A \square B)
 \end{array}$$

where $\bar{s}(f, g) = (f \square g)\lambda^{-1}$, $f \in \tau(I, A)$ and $g \in \tau(I, B)$.
 Let $t: A \square B \rightarrow A \times B$ be the continuous map defined by:



where p_A and p_B are the canonical projections.
 Since $Ut \cdot s = 1_{UA \times UB}$, it follows that s is an injection.

Let now h and k be two arbitrary maps from $A \square B$ to C , then

$$Uh \cdot s = Uk \cdot s \Rightarrow h = k.$$

In fact

$$Uh \cdot s = Uk \cdot s \Leftrightarrow h \cdot (f \square g) = k \cdot (f \square g) \Leftrightarrow [g, 1] \cdot \pi(h) \cdot f = [g, 1] \cdot \pi(k) \cdot f,$$

for any $f \in \tau(I, A)$ and $g \in \tau(I, B)$. Applying U , we obtain

$$(U\pi(h))(a) \cdot g = (U\pi(k))(a) \cdot g$$

for any $a \in UA$; and $g \in \tau(I, B)$. It follows that $U\pi(h) = U\pi(k)$, and then $h = k$.

Since $UA \times UB$ is in bijection with a subspace of $A \square B$, we can provide s with a structure of continuous map: then s is an epimorphism in Top and, being injective as we said above, it is a bijection in Set.

d) It follows from the continuity of $1_{UA \times UB}$

e) It is trivial.

f) It suffices to observe that the function $1_{UA \times UB} : A \otimes B \rightarrow A \square B$ is separately continuous.

g) If $f \in \tau(A \square B, C)$ then $f(a, -) : B \rightarrow C$ and $f(-, b) : A \rightarrow C$ are continuous for any $a \in UA$, $b \in UB$. Let consider now the function $U\pi(f) : UA \rightarrow U[B, C]$ applied to an arbitrary $a \in UA$. We have $(U\pi(f))(a) = \pi^{-1}(\pi(f) \cdot \bar{a}) \cdot \lambda^{-1} = f \cdot (\bar{a} \square 1) \cdot \lambda^{-1} = f(a, -)$

where $\bar{a} : I \rightarrow A$, $\bar{a}(*) = a$.

A similar proof applies to $\pi^{-1}g$, for any $g \in \tau(A, [B, C])$.

h) It follows from f) and g).

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