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A REMARK ON MONOIDAL CLOSED STRUCTURES ON TOP*

M.C. Pedicchio and F. Rossi

Introduction. The aim of this note is to characterize monoidal closed structures (in the sense of [2]) on the category Top of topological spaces and continuous maps.

We prove that any of such structures *must* satisfy the following conditions:

- the unit object is the singleton space;
- the tensor product has, as underlying set, the product set;
- the internal hom has, as underlying set, the set of continuous functions.

We heard that H. Niederle (see J. Činčura in [1]), in an unpublished paper, found similar results for some concrete categories, but, it seems that *symmetry is an essential condition* of his hypotheses.

Notation. By $\mathcal{U}$: Top $\to$ Set we denote the canonical forgetful functor.

For any A,B $\in$ Top :

- $\tau(A,B)$ stands for the set of continuous maps from A to B;
- $A\times B$ stands for the topological product;
- $A\circledast B$ stands for the product set $\mathcal{U}A\times\mathcal{U}B$, provided with the topology of separate continuity;
- $\langle A,B\rangle$ stands for the set $\tau(A,B)$ provided with the topology of pointwise convergence.

Proposition. *If* $(\square, I, \alpha, \lambda, \rho, [-,-])$ *is a monoidal closed structure (not necessarily symmetric) on* Top, *then*

a) $\mathcal{U}I = \{*\}$;

b) the underlying set of $[A,B]$ *is, up to natural isomorphisms* $\tau(A,B)$;

* "This paper is in final form and no version of it will be submitted for publication elsewhere".

c) the underlying set of $A \square B$ *is, up to natural isomorphisms* $\mathcal{U}A \times \mathcal{U}B$;

d) $A \times B \leqq A \square B$;

e) the isomorphisms α, λ, ρ *have canonical underlying functions;*

f) $A \square B \leqq A \otimes B$;

g) if $\pi : \tau(A \square B, C) \cong \tau(A, [B,C])$ *is the adjunction of the closed structure, then, for any* $f \in \tau(A \square B, C)$, πf *is defined by*

$$\pi f(a)(b) = f(a,b)\ , \quad a \in A,\ b \in B;$$

and, for any $g \in \tau(A, [B,C])$, $\pi^{-1} g$ *is defined by*

$$\pi^{-1} g(a,b) = g(a)(b)\ , \quad a \in A,\ b \in B;$$

h) $\langle A,B \rangle \leqq [A,B]$.

Proof. a) Since $\tau(I,I)$ is a commutative monoid, then card $\mathcal{U}I \leqq 1$. It is easy to see that it cannot be zero, and the result follows.
b) It suffices to recall that $\mathcal{U}$ is a representable functor, with representing object I.
c) Let $s\colon \mathcal{U}A \times \mathcal{U}B \to \mathcal{U}(A \square B)$ be the function defined by:

$$\begin{tikzcd}
\mathcal{U}A \times \mathcal{U}B \arrow[r, dashed, "s"] \arrow[d, phantom, "\cong" rotate=-90] & \mathcal{U}(A \square B) \arrow[d, phantom, "\cong" rotate=-90] \\
\tau(I,A) \times \tau(I,B) \arrow[r, "\bar{s}"] & \tau(I, A \square B)
\end{tikzcd}$$

where $\bar{s}(f,g) = (f \square g)\lambda^{-1}$, $f \in \tau(I,A)$ and $g \in \tau(I,B)$.
Let $t\colon A \square B \to A \times B$ be the continuous map defined by:

$$\begin{tikzcd}
 & A \square I \arrow[r, "\cong"] & A & \\
A \square B \arrow[ur, "1 \square !"] \arrow[dr, "! \square 1"'] \arrow[rrr, dashed, "t"] & & & A \times B \arrow[ul, "p_A"'] \arrow[dl, "p_B"] \\
 & I \square B \arrow[r, "\cong"'] & B &
\end{tikzcd}$$

where p_A and p_B are the canonical projections.
Since $\mathcal{U}t \cdot s = 1_{\mathcal{U}A \times \mathcal{U}B}$, it follows that s is an injection.

Let now h and k be two arbitrary maps from $A \square B$ to C, then

$$\mathcal{U}h \cdot s = \mathcal{U}k \cdot s \Rightarrow h = k.$$

In fact

$$\mathcal{U}h \cdot s = \mathcal{U}k \cdot s \Leftrightarrow h \cdot (f \square g) = k \cdot (f \square g) \Leftrightarrow [g,1] \cdot \pi(h) \cdot f = [g,1] \cdot \pi(k) \cdot f,$$

for any $f \in \tau(I,A)$ and $g \in \tau(I,B)$. Applying $\mathcal{U}$, we obtain

$$(\mathcal{U}\pi(h))(a) \cdot g = (\mathcal{U}\pi(k))(a) \cdot g$$

for any $a \in \mathcal{U}A$; and $g \in \tau(I,B)$. It follows that $\mathcal{U}\pi(h) = \mathcal{U}\pi(k)$, and then $h = k$.

Since $\mathcal{U}A \times \mathcal{U}B$ is in bijection with a subspace of $A \,\square\, B$, we can provide s with a structure of continuous map; then s is an epimorphism in Top and, being injective as we said above, it is a bijection in Set.

d) It follows from the continuity of $1_{\mathcal{U}A \times \mathcal{U}B}$

e) It is trivial.

f) It suffices to observe that the function $1_{\mathcal{U}A \times \mathcal{U}B}: A \otimes B \to A \,\square\, B$ is separately continuous.

g) If $f \in \tau(A \,\square\, B, C)$ then $f(a,-): B \to C$ and $f(-,b): A \to C$ are continuous for any $a \in \mathcal{U}A$, $b \in \mathcal{U}B$. Let consider now the function $\mathcal{U}\pi(f): \mathcal{U}A \to \mathcal{U}[B,C]$ applied to an arbitrary $a \in \mathcal{U}A$. We have $(\mathcal{U}\pi(f))(a) = \pi^{-1}(\pi(f)\cdot\bar{a})\cdot\lambda^{-1} = f\cdot(\bar{a} \,\square\, 1)\cdot\lambda^{-1} = f(a,-)$ where $\bar{a}: I \to A$, $\bar{a}(*) = a$.

A similar proof applies to $\pi^{-1}g$, for any $g \in \tau(A,[B,C])$.

h) It follows from f) and g).

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