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Jan Pelant; Svatopluk Poljak

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EXTENSIONS OF CYCLICALLY MONOTONE MAPPINGS

Jan Pelant and Svatopluk Poljak

The paper deals with cyclically monotone mappings and their subclass called strongly cyclically monotone mappings. We abbreviate them as c.m. and s.c.m., respectively. The c.m. mappings were introduced by R.T. Rockafellar [7] to characterize subgradients of convex functions. The s.c.m. mappings were obtained in [4] as a class of mappings having an interesting "periodical property" (see Theorems 5 and 6 below) when used as a part of a transition mapping of a discrete system. It was shown in [5] that some reasonable decision procedures when the decision depends on the impact of surrouding may be formulated as s.c.m. mappings. This application suggests a question whether each partial list of internally consistent decisions can be extended to some rule applicable to all impacts. We answer the question positively in Theorem 13 (of course only for the decisions identified with some s.c.m. mapping). This theorem is based on a statement which concerns the structure of subgradients and may be of some interest itself:

If $\partial(y) \cap \text{rel int } \partial(x)$ is non-empty, then $\partial(x) \subset \partial(y)$ where $\partial(x)$ and $\partial(y)$ are sets of all subgradients of a convex function u on R^n at points x and y, respectively.

We also survey other results on extensions of c.m. mappings. We give full proof to those which were only mentioned before.

Definition (Rockafellar [7]): Let $S \subset R^n$ and $s : S \to R^n$ be a multivalued mapping. Then s is called c.m. on S, if

$$(1) \qquad \sum_{i=1}^{k} (x_i - x_{i-1}) y_i \geq 0$$

for every $k \geq 2$, every $x_1, \dots, x_k = x_0$, and every choice of $y_i \in s(x_i)$, $i=1,\dots,k$.

It follows from the definition immediately that every restriction of a c.m. mapping is c.m. as well.

Let $u : R^n \to R$ be a convex function. The set of all subgradients of u at x is denoted by $\partial_u(x)$, i.e.

$$\partial_u(x) = \{ y \in R^n \mid \forall x' \quad u(x') - u(x) \geq (x'-x)y \} .$$

(We will omit the subscript u in $\partial(x)$ when it cannot make any confusion.)

Theorem 1 (Rockafellar [7]).

(i) The multivalued mapping $x \mapsto \partial_u(x)$ is c.m. for every convex function $u : R^n \to R$.

(ii) If a (multivalued) mapping $s : R^n \to R^n$ is c.m., then there is a convex function u such that $s(x) \subseteq \partial(x)$ for every x .

(iii) The function u in (ii) is unique (up to additive constant) provided s is maximal c.m. mapping.

As we deal with single-valued mappings only, we restrict the notion of c.m. as follows. We say that a mapping $f : S \to R^n$, $S \subset R^n$, is c.m. on S , if

$$(2) \qquad \sum_{i=1}^{k} (x_i - x_{i-1}) f(x_i) \geq 0$$

for all $k \geq 2$ and $x_1, \dots, x_k = x_0 \in S$.

Let us say a mapping $u : S \to R$ is a potential of f , if

$$(3) \qquad u(x) - u(y) \geq (x-y) f(y)$$

for all $x, y \in S$.

Parts (i) and (ii) of Theorem 1 remain true of we replace R^n by arbitrary (even finite) subset $S \subset R^n$.

Theorem 2 ([4]). Let $S \subset R^n$. A mapping $f : S \to R^n$ is c.m. if and only if it has some potential.

Sketch of the proof. If the mapping f has a potential, one can simply check that f is c.m. On the other hand, there are more possible ways of defining a potential for a given c.m. mapping. Rockafellar [7] used the formula

$$(4) \qquad u(x) = \sup \Big(\sum_{i=0}^{k} x_{i+1} f(x_i) - \sum_{i=0}^{k} x_i f(x_i) \Big)$$

where x_0 is a fixed element and the supremum is taken over all finite sequences $x_0, x_1, \dots, x_{k+1} = x \in S$. A Pultr [6] observed that a potential can be defined also by

(5) $$u(x) = \inf \left(\sum_{i=1}^{k+1} x_i f(x_i) - \sum_{i=0}^{k} x_i f(x_{i+1}) \right) ,$$

where again $x_0 \in S$ is fixed and the infimum is over the same set as above. Let us mention that the potentials defined by (4) and (5) may differ. (E.g. $S = \{(0,0),(1,1)\}$, $f(0,0) = (0,-1)$, $f(1,1) = (1,0)$.)

The next theorem says that every potential can be extended to a convex function.

Theorem 3 ([2]). Let $f : S \to R^n$, $S \subset R^n$, be c.m. and let u be a potential of f . Then there exists a convex function $\bar{u}$: conv $S \to R$ such that $\bar{u}|_S = u$ and $f(x) \in \partial_{\bar{u}}(x)$ for all $x \in S$. (conv S denotes the convex hull of S .)

Sketch of the proof. For every $y \in S$ consider a linear mapping $f_y : R^n \to R$ defined by

(6) $$f_y(x) = u(y) + (x-y)f(y) ,$$

and set $\bar{u}(x) = \sup \{ f_y(x) \mid y \in S \}$. The function $\bar{u}$ is convex as it is the supremum of convex functions f_y , and as $f_x(x) \geq f_y(x)$ according (3), we get $\bar{u}(x) = u(x)$, for all $x \in S$. Clearly $\bar{u}(x)$ must be finite on conv S .

Corollary 4. Every c.m. mapping f defined on $S \subset R^n$ can be extended to some c.m. mapping $\bar{f}$ defined on conv S .

Proof. Let $\bar{u}$ be the convex function defined in Theorem 3 (for arbitrary potential u), and $\bar{f}(x)$ be arbitrary subgradient of $\bar{u}$ at $x \in$ conv $S \setminus S$.

The c.m. mappings have certain interesting "periodical" properties which lead us to study them. E.g. the following holds.

Theorem 5 ([2]). Let A be a real symmetric matrix of size n and $f : R^n \to R^n$ be continuous c.m. Define a sequence $\{y_i\}$ by

(7) $$y_{i+1} = f(Ay_i)$$

for some initial y_0 . Then $\lim \| y_{i+2} - y_i \| = 0$ provided the sequence is bounded.

In case the c.m. mapping is defined only on a subset of R^n , the assumption on continuity must be replaced by another property.

We say that a c.m. mapping $f : S \to R^n$ is strongly c.m. on S , if

(8) $$\sum_{i=1}^{k}(x_i - x_{i-1})f(x_i) = 0 \Rightarrow f(x_1) = \dots = f(x_k)$$

for all $k \geq 2$ and every $x_1,\dots,x_k = x_0 \in S$.

Theorem 6 ([4]). Let A be a real symmetric matrix of size n and f be a s.c.m. mapping of finite range. Then the sequence (7) has period at most 2, i.e. $y_{k+2} = y_k$ for all k sufficiently large.

Theorem 6 has a very short proof, but it can be also derived from the more difficult Theorem 5. The reduction is based on the following extension result.

Theorem 7 ([2]). Let S be a finite subset of R^n and $f : S \to R^n$ be c.m. Then f can be extended to some continuous mapping $\bar{f} : R^n \to R^n$ if and only if f is s.c.m.

The additional condition (8) on c.m. mappings might look a bit restrictive, but we show that it is not the case. We can produce sufficiently many s.c.m. mappings due to Propositions 11, 12 and Theorem 13. Moreover, all the mappings with period at most 2 studied in [1] and [3] were s.c.m.

Lemma 8. Let $u : S \to R$ be a convex function. If $s \in \partial(y)$ and $u(x) - u(y) = (x-y)s$, then $s \in \partial(x)$.

Proof. Let $z \in S$ be arbitrary. As $s \in \partial(y)$, we have $u(z)-u(y) \geq (z-y)s$. Thus $u(z)-u(x) = (u(z)-u(y)) - (u(y)-u(x)) \geq$
$$\geq (z-y)s - (x-y)s = (z-x)s .$$
This proves that s is a subgradient of u at x.

Lemma 9. Let $f : S \to R^n$ be c.m. and u be a potential of f. Then the three following conditions are equivalent for every choice of $x_1,\dots,x_k = x_0 \in S$, $k \geq 2$.

(i) $\sum_{i=1}^{k}(x_i - x_{i-1})f(x_i) = 0$

(ii) $u(x_i) - u(x_{i-1}) = (x_i - x_{i-1})f(x_{i-1})$ for all $i=1,\dots,k$

(iii) $f(x_i), f(x_{i-1}) \in \partial(x_i)$ for all $i=1,\dots,k$.

Proof. Conditions (ii) and (iii) are equivalent due to Lemma 8. If (ii) holds, then

$$0 = \sum(u(x_i) - u(x_{i-1})) = \sum(x_i - x_{i-1})f(x_{i-1})$$

which gives (i). If (ii) does not hold, then $u(x_i) - u(x_{i-1}) >$

$> (x_i - x_{i-1})f(x_{i-1})$ for some i, while $u(x_i) - u(x_{i-1}) \geq$ $\geq (x_i - x_{i-1})f(x_{i-1})$ for all the remaining i's. Summing up These inequalities we get a contradiction to (i).

Proposition 10. Let $\prec$ be a good ordering of R^n, and u be a convex function on R^n. Then a mapping f defined by

$$(9) \qquad f(x) = \min_{\prec} \{ y \mid y \in \partial(x) \}$$

is strongly c.m.

Proof. Assume f is not s.c.m. Then there are some $k \geq 2$ and $x_1, \dots, x_k = x_0$ such that at least two $f(x_i)$'s are distinct and

$$\sum_{i=1}^{k} (x_i - x_{i-1}) f(x_i) = 0 .$$

Let $y = \min_{\prec} (f(x_1), \dots, f(x_k))$, and j be such that $f(x_{j-1}) = y \neq f(x_j)$. Using Lemma 9 (iii) we get $y = f(x_{j-1}) \in \partial(x_j)$. This is a contradiction with the choice of $f(x_j)$ as the minimum of $\partial(x_j)$ by (9).

Corollary 11 ([4]). Let u be a convex differentiable function on R^n. Then the gradient ∇u is s.c.m.

Proof. If u is differentiable, then $|\partial(x)| = 1$ for every x.

Though Proposition 10 proves existence of at least one s.c.m. mapping for every convex function, it does not give any explicit description of it.

Proposition 12. Let u be a convex function on R^n, and $\prec$ be the lexicographic order on R^n. Then the mapping f defined by formula (9) is s.c.m.

Proof. As the sets $\partial(x)$ are convex and compact, each of them contains the minimum element with respect $\prec$. The proof then proceeds as in Proposition 10.

Example. Consider the convex function $u(x) = u(x_1, \dots, x_n) = \max x_i$ and define a mapping $f(x_1, \dots, x_n) = (0, \dots, 0, 1, 0, \dots, 0)$ so that 1 is in the j-th position if $x_j \geq x_i$ for $i \leq j$ and $x_j > x_i$ for $i > j$. Then f is s.c.m. as it is the lexicographic minimal subgradient of u.

The main result of the paper is the following theorem.

Theorem 13. Let $f : S \to R^n$, $S \subset R^n$, be s.c.m. Then f can be extended to some s.c.m. mapping $\bar{f}$ defined on conv S.

Before proving Theorem 13 we need a lemma.

Lemma 14. Let u be a convex function on a convex set $S \subset R^n$, $s, t \in \partial(x)$ and α, β be positive reals with $\alpha + \beta = 1$. Then for every $y \in S$, $\alpha s + \beta t \in \partial(y)$ implies $s,t \in \partial(y)$.

Proof. As $s,t \in \partial(x)$, y must satisfy

(10) $$u(y) - u(x) \geqq (y-x)s \text{ , and}$$

(11) $$u(y) - u(x) \geqq (y-x)t \text{ .}$$

We claim that (10) and (11) hold with the equality, i.e.

(12) $$u(y) - u(x) = (y-x)s \text{ , and}$$

(13) $$u(y) - u(x) = (y-x)t \text{ .}$$

Assume that say (12) is not true, i.e.

(14) $$u(y) - u(x) > (y-x)s$$

(the opposite inequality is excluded by (10)).

Counting the linear combination of (14) and (11) with coeficients α and β respectively, we obtain

$$u(y) - u(x) > (y-x)(\alpha s + \beta t)$$

which contradicts to the assumption $\alpha s + \beta t \in \partial(y)$. It follows immediately from (12) and (13) that $s,t \in \partial(y)$.

Corollary 15. Let u be a convex function on $S \subset R^n$. If $\partial(y) \cap \cap \text{ rel int } \partial(x) \neq \emptyset$ for some x and y , then $\partial(x) \subset \partial(y)$.

Proof. Let $r \subset \partial(y) \cap \text{ rel int } \partial(x)$. Assume $s \in \partial(x)$, $s \neq r$ is given. As $\partial(x)$ is convex, there is some t so that r is an interior point of the segment with the endpoints s and t . Thus $s \in \partial(y)$ by Lemma 14.

Proof of Theorem 13. Let $u : \text{conv } S \rightarrow R$ be a convex function which is a potential to f . (Such a function u exists due to Theorem 3. Consider an auxiliar mapping D and its extension $\bar{D}$ which are defined as follows.

$$D : \{\partial(x) \mid x \in S\} \rightarrow R^n \text{ where } D(\partial(x)) = f(x) \text{ , and}$$
$$\bar{D} : \{\partial(x) \mid x \in \text{conv } S\} \rightarrow R^n \text{ where}$$
$$\bar{D}(\partial(x)) = \begin{cases} D(\partial(x)) & \text{if } \partial(x) \in \text{dom } D \\ \text{arbitrary } y \in \text{rel int } \partial(x) & \text{otherwise .} \end{cases}$$

The mapping D is defined correctly, as if $f(x) \neq f(x')$ and $\partial(x) = \partial(x')$ for some $x, x' \in S$, then f would not be s.c.m. by Lemma 9. Now, we can define $\bar{f}$ by $\bar{f}(x) = \bar{D}(\partial(x))$. Assume $\bar{f}$ is not s.c.m., and let $x_1, x_2, \ldots, x_k = x_0$ be the shortest sequence (with minimum k) such that $\sum(x_i - x_{i-1})f(x_i) = 0$ but not all $f(x_i)$'s are same. Using Lemma 9 (iii) we get

$$f(x_i), f(x_{i-1}) \in \partial(x_i) \quad \text{for all } i=1,\ldots,k . \tag{15}$$

We distinguish two cases a and b.

Case a. For every $i=1,\ldots,k$ there is some x_i' such that $x_i' \in S$, $\partial(x_i') = \partial(x_i)$. Using (15) and the definition of $\bar{f}$ we get

$$f(x_i'), f(x_{i-1}') \in \partial(x_i') \quad \text{for all } i .$$

Using Lemma 9 f is not s.c.m. which is a contradiction.

Case b. If Case a does not occur, then, say, $\partial(x_k) \notin \operatorname{dom} D$. Thus $\bar{f}(x_k) \in \operatorname{rel\,int} \partial(x_k)$, and as $\bar{f}(x_k) \in \partial(x_1)$ by (15), using Corollary 15 we set $\partial(x_k) \subset \partial(x_1)$. As $\bar{f}(x_{k-1}) \in \partial(x_k)$ by (15), it is $\bar{f}(x_{k-1}) \in \partial(x_1)$. Thus $x_1, \ldots, x_{k-1}$ form a smaller counterexample to (8).

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JAN PELANT, MATH.INSTITUT ČSAV, ŽITNÁ 25, 115 67 PRAHA 1
CZECHOSLOVAKIA
SVATOPLUK POLJAK, TECHNICAL UNIVERSITY, DEPT. OF SYSTEM ENG.,
THÁKUROVA 7, 166 29 PRAHA 6, CZECHOSLOVAKIA