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HORIZONTAL LIFT OF TENSOR FIEDS OF TYPE (1,1) FROM A MANIFOLD TO ITS TANGENT BUNDLE OF HIGHER ORDER

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INTRODUCTION

Let M be a manifold of dimension n, $P(M,G)$ be a principal fibre bundle and Γ be a connection in $P(M,G)$. Let $E = E(M,F,G,P)$ be the fibre bundle associated with $P(M,G)$ and with a standandard fibre F.

The connection Γ defines a horizontal lift of vector fidts from M to E. If X is a vector field on M, then we denote by X^H the horizontal lift of X to E with respect to Γ.

Let F be a tensor field of type (1,1) on M. We can define a tensor field $\tilde{F}$ of type (1,1) on E such that

$$\tilde{F}(X^H) = (FX)^H$$

for every vector field X on M. We will look for such a construction that the mapping $F \longrightarrow F$ has "nice" algebraic properties which permit us to prolong geometric structures from M to E.

This problem has been studied for several fibre bundles associated with the principal fibre bundle of linear frames - in these cases the the given connection has been a linear connection on M. In particular, K. Yano, S. Ishihara and E. M. Patterson studied this problem in the case of tangent and cotangent bundle [12], [13], J. Gancarzewicz and N. Rahmani in the case $E = T^*M \otimes TM$ [5] and N. Rahmani in the case $T^p_qM = T^*M \otimes \ldots \otimes T^*M \otimes TM \otimes \ldots \otimes TM$ [11]. The above problem was also studied by M. de Leon and M. Salgado [7] in the case of the fibre bundle of frames of order 2. (It is the unique case with a connection of higher order.)

In this paper we propose a solution of this problem in the case of the tangent bumdle of order r. The tangent bundle of order r will be denoted by T^rM.

This paper is in final form and no version of it will be submitted for publication elsewhere.

In Section I we recall main results about λ-lifts of functions and vector fields to the tangent bundle of order r.

In Sections II and III we study the horizontal lift of vector fields to T^rM and we characterize the brackets of vertical and horizontal vector fields - these results will be used in the next section.

A definition of a horizontal lifts of tensor fields of type (1,1) from M to T^rM will be proposed in Section IV. Also its algebraic properties will be studied. In the case of the tangent bundle $TM = T^1M$ our definition coincides with the definition due to K. Yano and S. Ishihara [12]. Next we use our construction to prolong some geometric structures (for example, almost complex, almost product, f-stuctures) from M to T^rM and we study the integrability of these prolonged structures. Our theorems generalize results of K. Yano and S. Ishihara obtained in the case of the tangent bundle [12].

I. PRELIMINARIES

Let M be a manifold and let r be a non-negative integer. We denote by T^rM the set of all r-jets at 0 of curves on M and let $\pi : T^rM \longrightarrow M$ be the target projection defined by

$$\pi(j_0^r\gamma) = \gamma(0)$$

Now $\pi : T^rM \longrightarrow M$ is a locally trivial fibre bundle associated with the principal fibre bundle F^rM of frames of order r and with the standard fibre $\mathbb{R}^{nr}$, where $n = \dim M$.

If (U, x^i) is a chart on M we denote by

$$\{\pi^{-1}(U), x^{i,\lambda} \ : i = 1,\dots,n, \ \lambda = 0,\dots,r\}$$

the induced chart on T^rM defined by

$$x^{i,\lambda}(j_0^r\gamma) = \frac{1}{\lambda!}\frac{d^\lambda}{dt^\lambda}(x^i \circ \gamma)(0) \quad . \tag{1.1}$$

For every $\lambda = 0,\dots,r$ and every function f of class C^∞ on M we define the function $f^{(\lambda)}$ on T^rM by the formula (see [10])

$$f^{(\lambda)}(j_0^r\gamma) = \frac{1}{\lambda!}\frac{d^\lambda}{dt^\lambda}(f \circ \gamma)(0) \quad .$$

The function $f^{(\lambda)}$ is called λ-lift of f from M to T^rM. It is of class C^∞ on T^rM. The 0-lift $f^{(0)} = f \circ \pi$ is also called the vertical lift. We set $f^{(\lambda)} = 0$ if λ is negative.

If (U, x^i) is a chart on M, then for the induced chart defined by (1.1) we have

$$x^{i,\lambda} = (x^i)^{(\lambda)} , \tag{1.2}$$

for $i = 1, \ldots, n$ and $\lambda = 0, \ldots, r$. It is easy to verify (see Lemma 1.2 [10]) that

$$(af + bg)^{(\lambda)} = a\, f^{(\lambda)} + b\, g^{(\lambda)} \tag{1.3}$$

$$(fg)^{(\lambda)} = \sum_{\mu=0}^{\lambda} f^{(\mu)} g^{(\lambda - \mu)} \tag{1.4}$$

for all functions f, g on M and all real numbers a, b.

The family of functions $f^{(\lambda)}$ is important because vector fields on T^rM are characterized by their action on functions of type $f^{(\lambda)}$. More precisely, we have:

<u>PROPOSITION 1.1</u>. (see [10]) If $\tilde{X}$ and $\tilde{Y}$ are two vector fields on T^rM such that $\tilde{X}(f^{(\lambda)}) = \tilde{Y}(f^{(\lambda)})$ for every function f on M and $\lambda = 0, \ldots, r$, then $\tilde{X} = \tilde{Y}$.

<u>Proof</u>. If (U, x^i) is a chart on M, then by (1.2) we have

$$\tilde{X}(x^{i,\lambda}) = \tilde{Y}(x^{i,\lambda})$$

for the induced chart, $i = 1, \ldots, n$ and $\lambda = 0, \ldots, r$. Thus $\tilde{X} = \tilde{Y}$ on $\pi^{-1}(U)$.

A. Morimoto defined in [10] the λ-lift of X to T^rM for any vector field X on M and $\lambda = 0, \ldots, r$. These lifts were defined by the following proposition:

<u>PROPOSITION 1.2</u>. (see [10]) If X is a vector field on M and $\lambda = 0, \ldots, r$, then there exists one and only one vector field $X^{(\lambda)}$ on T^rM such that

$$X^{(\lambda)}(f^{(\mu)}) = (Xf)^{(\lambda + \mu - r)} \tag{1.5}$$

for all functions f on M and $\mu = 0, \ldots, r$.

This unique vector field $X^{(\lambda)}$ is called the λ-lift of X to T^rM. For $\lambda < 0$ we define $X^{(\lambda)} = 0$.

A vector field $\tilde{X}$ on T^rM is vertical if and only if $\tilde{X}(f^{(0)}) = 0$ for every function f on M. By virtue of this remark and by Proposition 1.2 $X^{(\lambda)}$ is a vertical vector field on T^rM for each vector field X on M and $\lambda = 0,\dots,r-1$.

According to Propositions 1.1 and 1.2 it is easy to check

$$(1.6)\qquad \begin{cases} (fX)^{(\lambda)} = \sum_{\mu=0}^{\lambda} f^{(\mu)} X^{(\lambda-\mu)} \\ [X^{(\lambda)}, Y^{(\mu)}] = [X,Y]^{(\lambda+\mu-r)} \end{cases}$$

where X, Y are vector fields and f is a function on M (see [10]).

If (U,x^i) is a chart on U and we denote by

$$\left\{\frac{\partial}{\partial x^i}\right\}_{i=1,\dots,n}, \quad \left\{\frac{\partial}{\partial x^{i,\nu}}\right\}_{i=1,\dots,n;\nu=0,\dots,r}$$

the canonical frames associated with (U,x^i) and with the induced chart $(\pi^{-1}(U), x^{i,\nu})$ respectively, then using (1.5), (1.2) and Proposition 1.1 we obtain

$$(1.7)\qquad \frac{\partial}{\partial x^{i,\nu}} = \left(\frac{\partial}{\partial x^i}\right)^{(r-\lambda)}$$

By using (1.7) and Proposition 1.2 we obtain:

<u>PROPOSITION 1.3.</u> (see Lemma 1.4 [10]) If X is a vector field on M and $X = X^i \frac{\partial}{\partial x^i}$ on U, then

$$X^{(\lambda)} = \sum_{\mu=r-\lambda}^{r} (X^i)^{(\lambda+\mu-r)} \frac{\partial}{\partial x^{i,\mu}}$$

on $\pi^{-1}(U)$.

<u>II. CONNECTIONS OF ORDER r AND HORIZONTAL LIFTS OF VECTOR FIELDS</u>

Let M be a manifold of dimension n. We denote by F^rM the set of all r-jets at $0 \in \mathbb{R}^n$ of local diffeomorphisms of neighbourhoods into M. Let $\pi: F^rM \longrightarrow M$ be the target projection defined by

$$\pi(j_0^r\varphi) = \varphi(0) \quad .$$

F^rM is a principal fibre bundle with the structural group L_n^r, where L_n^r is the Lie group of all r-jets at 0 of local diffeomorphisms ξ of $\mathbb{R}^n$ such that $\xi(0) = 0$. The action of L_n^r on F^rM is given by the formula

$$j_0^r\varphi \cdot j_0^r\xi = j_0^r(\varphi\circ\xi) \quad .$$

For $s < r$ we define $\pi_s^r: F^rM \longrightarrow F^sM$ by $\pi_s^r(j^r\varphi) = j^s\varphi$. The projection π_s^r is a homomorphism of principal fibre bundles.

We denote by $J^r_0(\mathbf{R},\mathbf{R}^n)_0$ the set of all r-jets at 0 of mappings $h:\mathbf{R} \longrightarrow \mathbf{R}^n$ such that $h(0) = 0$. The group L^r_n acts on $J^r_0(\mathbf{R},\mathbf{R}^n)_0$ on the left as follows:

$$j^r_0\xi \cdot j^r_0 h = j^r_0(\xi \circ h) .$$

Let $E = F^rM \times J^r_0(\mathbf{R},\mathbf{R}^n)_0/\sim$ be the associated fibre bundle, that is, E is the quotient set of $F^rM \times J^r_0(\mathbf{R},\mathbf{R}^n)_0$ by the equivalence relation $\sim$, where $\sim$ is defined in the following way:

$$(p,z) \sim (p',z') \iff \exists\ \xi \in L^r_n :\ p' = p\cdot\xi ,\ z' = \xi^{-1}\cdot z .$$

We denote by $\Phi: F^rM \times J^r_0(\mathbf{R},\mathbf{R}^n)_0 \longrightarrow E$ the canonical projection, i.e. $\Phi(p,z)$ is the equivalence class of (p,z). Let $\pi_E : E \longrightarrow M$ be the projection given by $\pi_E(\Phi(p,z)) = \pi(p)$. The associated fibre bundle E is isomorphic to T^rM – the isomorphism $\varkappa : E \longrightarrow T^rM$ is defined by:

$$\varkappa(\Phi(j^r_0\varphi, j^r_0 h)) = j^r_0(\varphi \circ h) .$$

The composition $\varkappa\circ\Phi$ will be also denoted by Φ.

Let l^r_n be the Lie algebra of L^r_n. l^r_n is a space of r-jets at 0 of mappings $X:\mathbf{R}^n \longrightarrow \mathbf{R}^n$ such that $X(0) = 0$ and the bracket is given by:

$$[j^r_0X, j^r_0Y] = j^r_0(X\circ Y - Y\circ X) .$$

We set $e_i^{i_1\dots i_s} = j^r_0(X_i^{i_1\dots i_s})$, where

$$X_i^{i_1\dots i_s}(u^1,\dots,u^n) = (0,\dots,0,\underbrace{u^{i_1}\dots u^{i_s}}_{(i)},0,\dots,0) .$$

The family

$$\left\{ e_i^{i_1\dots i_s} :\ i=1,\dots,n;\ 1\le i_1 \le \dots \le i_s \le n;\ s=1,\dots,r \right\}$$

is a base of l^r_n.

Let $\Gamma^{(r)}$ be a connection in F^rM. Such a connection is called a connection of order r on M. We denote by ω the connection form of $\Gamma^{(r)}$. The form ω is an l^r_n-valued 1-form on F^rM. If (U,φ), $\varphi = (x^i)$, is a chart on M, then we denote by σ_φ the section of $F^rM|U$ defined by

$$\sigma_\varphi(x) = j^r(\varphi^{-1}\circ\ \tau_{\varphi(x)}) ,$$

where $\tau_{\varphi(x)}:\mathbf{R}^n \longrightarrow \mathbf{R}^n$ is the translation. Now there exists one and only one family of functions

$$\left\{\Gamma^i_{kj_1\dots j_s} : i,k,j_\alpha=1,\dots,n;\ \alpha=1,\dots,s;\ s=1,\dots,r\right\}$$

of class C^∞ on U such that

$$(\mathfrak{S}^*_\varphi\omega)\left(\frac{\partial}{\partial x^k}\right) = \sum_{i=1}^{n}\sum_{s=1}^{r}\sum_{1\le j_1\le\dots\le j_s\le n}\Gamma^i_{kj_1\dots j_s}\, e_i^{j_1\dots j_s}$$

(2.1)

$$= \sum_{s=1}^{r}\frac{1}{s!}\Gamma^i_{kj_1\dots j_s}\, e_i^{j_1\dots j_s}$$

and $\Gamma^i_{kj_1\dots j_s}$ are symmetric with respect to $(j_1,\dots,j_s)$. These functions are called coordinates of $\Gamma^{(r)}$ with respect to (U,φ).

A connection $\Gamma^{(r)}$ of order r on M determines a decomposition

$$T(T^rM) = V(T^rM)\oplus H\ ,$$

where $V(T^rM)$ denotes the fibre bundle of vertical vectors on T^rM. Hence, for any point y of T^rM $d_y\pi|H_y : H_y \longrightarrow T_{\pi(y)}M$ is an isomorphism. If X is a vector field on M, then we define the horizontallift X^H of X to T^rM by the following formula:

$$X^H_y = (d_y\pi\,|\,H_y)^{-1}(X_{\pi(y)}) \quad . \tag{2.2}$$

If (U,x^i) is a chart on M, then we can prove by the straightforward calculation that X^H has the followin coordinates

$$\begin{cases} X^{i,0} = X^i \\ X^{i,\nu} = \sum_{k=1}^{r}\ \sum_{\mu_1+\dots+\mu_k=\nu}\frac{1}{k!}X^j\,\Gamma^i_{ji_1\dots i_k}\,x^{i_1,\mu_1}\dots x^{i_k,\mu_k}\end{cases} \tag{2.3}$$

$(\nu = 1,\dots,r)$ with respect to the induced chart, where

$$X = X^i\frac{\partial}{\partial x^i}\ ,\quad X^H = X^{i,\nu}\frac{\partial}{\partial x^{i,\nu}}\quad .$$

The following proposition is an immediade consequence of (2.2).

PROPOSITION 2.1. If X, Y are vector fields on M and f is a function on M, then

$$(X+Y)^H = X^H + Y^H\ ,\quad (fX)^H = f^{(0)}X^H\ .$$

III. A CHARACTERIZATION OF BRACKETS OF VERTICAL AND HORIZONTAL VECTOR FIELDS

At first, we characterize the bracket $[X^H,Y^H]$, where X and Y are vector fields on M.

Let p be a point of F^rM and $x = \pi(p)$. The mapping

$$p: J_0^r(\mathbf{R},\mathbf{R}^n)_0 \ni z \longrightarrow \phi(p,z) \in T_x^rM = \pi^{-1}(x)$$

(which will be also denoted by p) gives a diffeomorphism between $J_0^r(R,R^n)_0$ and T_x^rM. For any $y = \phi(p,z)$ we define

$$\psi_{(p,y)}: L_n^r \ni \xi \longrightarrow p(\xi\cdot p^{-1}(y)) \in T_x^rM \quad .$$

If X, Y are vector fields on M and y is a point of T^rM, we set

$$R^{\square}(X,Y)(y) \;=\; -2\, d_e\psi_{(p,y)}(\Omega_p(X_p^{\Gamma},Y_p^{\Gamma})) \tag{3.1}$$

where p is a point of F^rM such that $\pi(p) = \pi(y)$, Ω is the curvature form of $\Gamma^{(r)}$ and X^{Γ}, Y^{Γ} are the horizontal lifts of X, Y to F^rM with respect to $\Gamma^{(r)}$. We have:

LEMMA 3.1. $R^{\square}(X,Y)(y)$ is a vector of $T_y(T^rM)$ which is independent of the choice of p such that $\pi(p) = \pi(y)$.

PROOF. Let p' be another point of F^rM such that $\pi(p') = \pi(p)$ and let z,z' be two elements of $J_0^r(\mathbf{R},\mathbf{R}^n)_0$ satysfying the formula $y = \phi(p,z) = \phi(p',z')$. Now, there is $\eta \in L_n^r$ such that $p' = p\cdot\eta$, $z' = \eta^{-1}\cdot z$. Thus

$$\begin{aligned}
\psi_{(p',y)}(\xi) &= p'(\xi\cdot(p')^{-1}(y)) \\
&= p\xi(\xi\eta^{-1}\cdot p^{-1}(y)) \\
&= p(\eta\xi\eta^{-1}\cdot p^{-1}(y)) \\
&= (\psi_{(p,y)}\circ \mathrm{ad}_{\eta})(\xi) \ .
\end{aligned}$$

We know that Ω is a tensorial form of type Ad L_n^r, that is,

$$(R_{\xi})^{*}\Omega = \mathrm{Ad}_{\xi^{-1}}\cdot\Omega \ .$$

On the other hand, X^{Γ} and Y^{Γ} are invariant vector fields on F^rM, i.e.

$$dR_{\xi}(X_p^{\Gamma}) = X_{p'}^{\Gamma}, \qquad dR_{\xi}(Y_p^{\Gamma}) = Y_{p'}^{\Gamma} \ .$$

According to the above remarks we have

$$\begin{aligned} d_e\Psi(p',y)(\Omega_{p'}(X^\Gamma_{p'},Y^\Gamma_{p'})) &= d_e\Psi(p',y)(\Omega_{p'}(dR_\xi(X^\Gamma_p),dR_\xi(Y^\Gamma_p)) \\ &= (d_e\Psi(p,y)\circ \mathrm{Ad}_\xi)((R^*_\xi\Omega)_p(X^\Gamma_p,Y^\Gamma_p)) \\ &= (d_e\Psi(p,y)\circ \mathrm{Ad}_\xi)(\mathrm{Ad}_{\xi^{-1}}(\Omega_p(X^\Gamma_p,Y^\Gamma_p))) \\ &= d_e\Psi(p,y)(\Omega_p(X^\Gamma_p,Y^\Gamma_p)) \quad . \end{aligned}$$

Our proof is completed.

Now we can formulate the following proposition:

PROPOSITION 3.2. If X and Y are vector fields on M, then

$$[X^H,Y^H] = [X,Y]^H + R^\Omega(X,Y) \quad .$$

PROOF. Since X^H is a projectable vector fields and X is its projection it is sufficient to show that

$$\text{(3.2)} \qquad v([X^H,Y^H]) = R^\Omega(X,Y) \quad ,$$

where $v([X^H,Y^H])$ denotes the vertical component of $[X^H,Y^H]$. At first, we observe

$$\begin{aligned} \Omega(X^\Gamma,Y^\Gamma) &= d\omega(X^\Gamma,Y^\Gamma) \\ \text{(3.3)} \qquad &= \tfrac{1}{2}\left\{X^\Gamma(\omega(Y^\Gamma)) - Y^\Gamma(\omega(X^\Gamma)) - \omega([X^\Gamma,Y^\Gamma])\right\} \\ &= -\tfrac{1}{2}\,\omega(v[X^\Gamma,Y^\Gamma]) \quad . \end{aligned}$$

On the other hand, if V is a vertical vector of $T_p(F^rM)$, then

$$\text{(3.4)} \qquad \omega(V) = (d_e\varrho_p)^{-1}(V) \quad ,$$

where ϱ_p is the mapping defined by

$$\varrho_p : L^r \ni \xi \longrightarrow p\cdot\xi \in F^rM \quad .$$

Let $y = \Phi(p,z)$. By using (3.1), (3.3) and (3.4) we have

$$R^\Omega(X,Y)(y) = -2\, d_e\Psi(p,y)(\Omega_p(X^\Gamma_p,Y^\Gamma_p))$$

$$= (d_e\Psi_{(p,y)} \circ (d_e\rho_p)^{-1})(v([X^\Gamma,Y^\Gamma](p))) \quad .$$

Next, by using the formula

$$\Psi_{(p,y)} \circ \rho_p^{-1} = \Phi_z|F_x^rM \quad , \quad x = \pi(p) \quad ,$$

where $\Phi_z : F^rM \longrightarrow T^rM$, $\Phi_z(\_) = \Phi(\_,z)$ and $F_x^rM = \pi^{-1}(x)$, we obtain

$$\begin{aligned} R^\Box(X,Y)(y) &= d_p\Phi_z(v([X^\Gamma,Y^\Gamma](p))) \\ &= v(d_p\Phi_z([X^\Gamma,Y^\Gamma](p)) \\ &= v([X^H,Y^H](\Phi_z(p))) \\ &= v([X^H,Y^H](y)) \end{aligned}$$

because the vector fields X^Γ (resp. Y^Γ) and X^H (resp. Y^H) are Φ_z-conjugate, that is, $d_p\Phi_z(X^\Gamma(p)) = X^H(\Phi_z(p))$ (resp. $d_p\Phi_z(Y^\Gamma(p)) = Y^H(\Phi_z(p))$).

REMARK. It is easy to observe that Proposition 3.2 is true for any fibre bundle associated with any principal fibre bundle with a connection. In the general case we use exactely the same arguments.

In order to calculate $[X^H,Y^{(\lambda)}]$ for $\lambda = 0,\dots,r-1$, we introduce the following notations.

If x is a point of M, then we denote by $J_x^\lambda(TM)$ the space of all λ-jets at x of vector fields on M. Let

$$J^\lambda(TM) = \bigcup_x J_x^\lambda(TM)$$

be the λ-jet prolongation of the tangent bundle. We denote by $\rho : J^\lambda(TM) \longrightarrow M$ the projection defined by $\rho(j_x^\lambda X) = x$. Now, $\rho : J^\lambda(TM) \longrightarrow M$ is a vector bundle associated with $F^{\lambda+1}M$. If (U,x^i) is a chart on M, then the induced chart

$$\{\rho^{-1}(U), x^i, W^i, W^i_{i_1\dots i_s} : i,i_1,\dots,i_s = 1,\dots,n;\ s = 1,\dots,\lambda\}$$

on $J^\lambda(TM)$ is given by the following formulas:

$$x^i(j_xW) = x^i(x)$$

$$W^i(j_xW) = W^i(x)$$

$$w^i_{i_1\ldots i_s}(j^\lambda_x W) = \left(\frac{\partial^s}{\partial x^{i_1}\ldots\partial x^{i_s}} W^i\right)(x) \quad .$$

(The functions $w^i_{i_1\ldots i_s}$ are symmetric with respect to $i_1,\ldots,i_s$.)

If $\eta \leq \lambda$, then we define $\rho^\lambda_\eta : J^\lambda(TM) \longrightarrow J^\eta(TM)$ by $\rho^\lambda_\eta(j^\lambda_x W) = j^\eta_x W$. The mapping ρ^λ_η is a homomorphism of vector bundles. If X is a vector field on M, we denote by $J^\lambda X$ the section of $J^\lambda(TM)$ given by the formula:

$$(3.6) \qquad (J^\lambda X)(x) = j^\lambda_x X \quad .$$

Let $\underline{J^\lambda(TM)}$ be the space of all sections of $J^\lambda(TM)$. If σ is an element of $\underline{J^\lambda(TM)}$, $\lambda < r$, then we consider the vector field $\sigma^{(\lambda)}$ on T^rM defined by

$$(3.7) \qquad \sigma^{(\lambda)}(y) = Z^{(\lambda)}(y) \quad ,$$

where Z is a vector field on M such that $\sigma(\pi(y)) = j^\lambda_{\pi(y)} Z$. The vector field $\sigma^{(\lambda)}$ is well-defined because the λ-lift $Z^{(\lambda)}(y)$ depends only on $j_{\pi(y)} Z$ (see Proposition 1.3). It is clear that $\sigma^{(\lambda)}$ is a vertical vector field on T^rM $(\lambda < r)$. If (U, x^i) is a chart on M and we denote

$$\sigma^i = w^i \circ \sigma \quad , \quad \sigma^i_{i_1\ldots i_s} = w^i_{i_1\ldots i_s} \circ \sigma$$

for the induced chart on $J^\lambda(TM)$, then for the vector field

$$\sigma^{(\lambda)} = \sigma^{i,\nu} \frac{\partial}{\partial x^{i,\nu}}$$

we have the following local expression

$$(3.8) \qquad \sigma^{i,\nu} = \begin{cases} 0 & \text{if } \nu < r-\lambda \\ \sigma^i & \text{if } \nu = r-\lambda \\ \displaystyle\sum_{s=1}^{\nu+\lambda-r} \sum_{\mu_1+\ldots+\mu_s=\nu+\lambda-r} \frac{1}{s!} \sigma^i_{i_1\ldots i_s} x^{i_1,\mu_1}\ldots x^{i_s,\mu_s} & \text{if } \nu > r-\lambda \end{cases}$$

Using (3.7) and (1.4) it is easy to prove the following proposition:

<u>PROPOSITION 3.4</u>. If σ, σ' are sections of $J^\lambda(TM)$, f is a function on M and a, a' are real numbers, then

$$(a\sigma + a'\sigma')^{(\lambda)} = a\,\sigma^{(\lambda)} + a'\,\sigma'^{(\lambda)} ,$$

$$(f\sigma)^{(\lambda)} = \sum_{\mu=0}^{\lambda} f^{(\mu)}\,\sigma^{(\lambda-\mu)} .$$

Since $\pi^r_\lambda : F^r M \longrightarrow F^\lambda M$ is a homomorphism of principal fibre bundles for $\lambda \leqslant r$, a given connection $\Gamma^{(r)}$ of order r on M induces a connection $\Gamma^{(\lambda)}$ in F M (a connection of order λ on M). The bundle $J^\lambda(TM)$ is associated with $F^{\lambda+1}M$, thus for $\lambda < r$, $\Gamma^{(\lambda+1)}$ defines the covariant derivation $\overset{(\lambda+1)}{\nabla}$ of sections of $J^\lambda(TM)$ (see [1])

$$\overset{(\lambda+1)}{\nabla} : \mathfrak{X}(M) \times \underline{J^\lambda(TM)} \ni (X,\sigma) \longrightarrow \overset{(\lambda+1)}{\nabla}_X \sigma \in \underline{J^\lambda(TM)}$$

$$(3.9) \qquad (\overset{(\lambda+1)}{\nabla}_X \sigma)(x) = I_{\sigma(x)}((X^H \circ \sigma - d\sigma \circ X)(x)) ,$$

where X^H denotes the horizontal lift of a vector field X to $J^\lambda(TM)$ with respect to $\overset{(\lambda+1)}{\nabla}$ and $I_{\sigma(x)}$ is the natural isomorphism between the vector spaces $T_{\sigma(x)}(J^\lambda_x(TM))$ and $J^\lambda_x(TM)$ (we must observe that $(X^H \circ \sigma - d\sigma \circ X)(x)$ is a vertical vector).

The main proposition of this section is the following one:

<u>PROPOSITION 3.5.</u> If X, Y are vector fields on M and $\lambda = 0,\dots,r-1$, then

$$(3.10) \qquad [X^H, Y^{(\lambda)}] = (\overset{(\lambda+1)}{\nabla}_X J^\lambda Y)^{(\lambda)} ,$$

where $J^\lambda Y$ is the section defined by (3.6).

<u>PROOF.</u> We show the formula (3.10) for r = 2 (to simplify the calculations).

At first, we assume $\lambda = 0$. If $X = X^i\, \partial/\partial x^i$ and $Y = Y^i\, \partial/\partial x^i$, then using (2.3) and Proposition 1.3 we obtain

$$(3.11) \qquad [X^H, Y^{(0)}] = \left\{ X^k \frac{\partial Y^i}{\partial x^k} + \Gamma^i_{kj}\, X^k\, Y^j \right\} \frac{\partial}{\partial x^{i,2}} .$$

On the other hand, using (3.9) we can calculate the local expression of $\overset{(1)}{\nabla}_X Y$ with respect to the induced chart on $J^0(TM) = TM$

$$W^i(\overset{(1)}{\nabla}_X Y) = X^k \frac{\partial Y^i}{\partial x^k} + \Gamma^i_{kj}\, X^k\, Y^j ,$$

and next, by using (3.8) we obtain

$$(3.12) \qquad (\overset{(1)}{\nabla}_X Y)^{(0)} = \left\{ X^k \frac{\partial Y^i}{\partial x^k} + \Gamma^i_{kj}\, X^k\, Y^j \right\} \frac{\partial}{\partial x^{i,2}}$$

The formulas (3.11) and (3.12) prove our proposition for $\;\; = 0$ and

$r = 2$.

Secondly, if $\lambda = 1$, then by using (2.3) and Proposition 1.3 we obtain:

$$
(3.13)\quad \left\{
\begin{aligned}
[X^H, Y^{(1)}] &= X^j \left\{ \frac{\partial Y^i}{\partial x^j} + \Gamma^i_{jk} Y^k \right\} \frac{\partial}{\partial x^{i,1}} + \\
&+ \left\{ (X^j \frac{\partial Y^i}{\partial x^j})^{(1)} - X^j \Gamma^k_{js} \frac{\partial Y^i}{\partial x^k} x^{s,1} + \right. \\
&\left. + X^j \Gamma^i_{js} (Y^s)^{(1)} + X^j \Gamma^i_{jks} Y^k x^{s,1} \right\} \frac{\partial}{\partial x^{s,2}} .
\end{aligned}
\right.
$$

On the other hand, using (3.9) and (3.6) we can calculate the local expression of $\overset{(2)}{\nabla}_X J^1 Y$ with respect to the induced chart on $J^1(TM)$

$$
W^i(\overset{(2)}{\nabla}_X J^1 Y) = X^j \left\{ \frac{\partial Y^i}{\partial x^j} + \Gamma^i_{jk} Y^k \right\}
$$

$$
W^i_k(\overset{(2)}{\nabla}_X J^1 Y) = X^j \left\{ \frac{\partial^2 Y^i}{\partial x^j \partial x^k} - \Gamma^s_{jk} \frac{\partial Y^i}{\partial x^s} + \Gamma^i_{js} \frac{\partial Y^s}{\partial x^k} + \Gamma^i_{jks} Y^s \right\}
$$

Now, according to (3.8) we have

$$
(3.14)\quad \left\{
\begin{aligned}
(\overset{(2)}{\nabla}_X J^1 Y)^{(1)} &= W^i(\overset{(2)}{\nabla}_X J^1 Y) \frac{\partial}{\partial x^{i,1}} + W^i_k(\overset{(2)}{\nabla}_X J^1 Y)\, x^{k,1} \frac{\partial}{\partial x^{i,2}} \\
&= X^j \left\{ \frac{\partial Y^i}{\partial x^j} + \Gamma^i_{jk} Y^k \right\} \frac{\partial}{\partial x^{i,1}} + \\
&\quad + X^j \left\{ \frac{\partial^2 Y^i}{\partial x^j \partial x^k} - \Gamma^s_{jk} \frac{\partial Y^i}{\partial x^s} + \Gamma^i_{js} \frac{\partial Y^s}{\partial x^k} + \right. \\
&\quad \left. + \Gamma^i_{jks} Y^s \right\} x^{k,1} \frac{\partial}{\partial x^{i,2}} \\
&= \left\{ X^j \frac{\partial Y^i}{\partial x^j} + \Gamma^i_{jk} X^j Y^k \right\} \frac{\partial}{\partial x^{i,1}} + \\
&\quad + \left\{ (X^j \frac{\partial Y^i}{\partial x^j})^{(1)} - X^j \Gamma^s_{jk} \frac{\partial Y^i}{\partial x^s} x^{k,1} + \right. \\
&\quad \left. + X^j \Gamma^i_{js} (Y^s)^{(1)} + X^j \Gamma^i_{jks} Y^s x^{k,1} \right\} \frac{\partial}{\partial x^{i,2}} .
\end{aligned}
\right.
$$

The formulas (3.13) and (3.14) show our proposition for $\lambda = 1$ and $r = 2$.

The proof is completed. (For simplicity we presented the calculation only for $r = 2$.)

IV. HORIZONTAL LIFTING OF TENSOR FIELDS OF TYPE (1,1) TO T^rM

We propose the following definition:

DEFINITION 4.1. Let F be a tensor field of type (1,1) on M.

A tensor field F^H of type (1.1) on T^rM is called a horizontal lift of F to T^rM if

$$(4.1) \qquad F^H(X^H) = (FX)^H \quad , \quad F^H(X^{(\lambda)}) = (FX)^{(\lambda)}$$

for every vector field X on M and $\lambda = 0,\ldots,r-1$.

We observe that according to Section I there is one and only one tensor field F^H on T^rM satisfying the formulas (4.1). Definition 4.1 implies immediately:

PROPOSITION 4.2. If F, G are tensor fields of type (1.1) on M and a, b are real numbers, then

$$(aF + bG)^H = a\,F^H + b\,G^H$$

$$(F \circ G)^H = F^H \circ G^H$$

$$(I_M)^H = I_{T^rM} \quad ,$$

where I_M and I_{T^rM} are the identity tensor fields of type (1.1) on M and T^rM respectively. In Particular, if P is a polynomial with constant real coefficients, then for any tensor field F of type (1.1) on M we have

$$P(F^H) = (P(F))^H \quad .$$

The following corollary is an immediate consequence of Proposition 4.2.

COROLLARY 4.3. If F is an almost complex structure (resp. an almost product structure, an f-structure) on M, then F^H is an almost complex structure (resp. an almost product stucture, an f-structure) on T^rM.

To study the integrability of geometric stuctures of type F^H we will compute the Nijenhuis tensor of F^H. Before formulating our proposition about Nijenhuis tensor of F^H we introduce the following notation. If F is a tensor field of type (1.1) on M and σ is a section of $J^\lambda(TM)$, we define a new section $F\sigma$ of $J^\lambda(TM)$ by the formula

$$(4.2) \qquad (F\sigma)(x) = j_x(FX) \quad ,$$

where X is a vector field on M such that $j_xX = \sigma(x)$. It is clear that $F\sigma$ is a well-defined section of J (TM).

Now we have:

PROPOSITION 4.4. Let F be a tensor field of type (1.1) on M. If X, Y are vector fields on M and λ, $\eta = 0,\dots,r-1$, then

$$N_{F^H}(X^H,Y^H) = (N_F(X,Y))^H + R^{\square}(FX,FY) + (F^H)^2(R^{\square}(X,Y)) - \tag{4.3}$$
$$- F^H(\, R^{\square}(FX,Y) + R^{\square}(X,FY)\,)$$

$$N_{F^H}(X^H,Y^{(\lambda)}) = \Big\{ \overset{(\lambda+1)}{\nabla}_{FX} J^{\lambda} FY - F(\overset{(\lambda+1)}{\nabla}_{FX} J^{\lambda} Y) + F^2(\overset{(\lambda+1)}{\nabla}_{X} J^{\lambda} Y) - \tag{4.4}$$
$$- F(\overset{(\lambda+1)}{\nabla}_{X} J^{\lambda} FY) \Big\}^{(\lambda)}$$

$$N_{F^H}(X^{(\lambda)},Y^{(\eta)}) = (N_F(X,Y))^{(\lambda+\eta-r)} \quad , \tag{4.5}$$

where N_F and N_{F^H} denote the Nijenhuis tensors of F and F^H respectively.

PROOF. The formulas (4.3) and (4.5) are consequences of Proposition 3.2 and (1.6). The formula (4.4) follows from Proposition 3.5 and from the formula

$$F^H \sigma^{(\lambda)} = (F\sigma)^{(\lambda)} \quad , \tag{4.6}$$

where σ is a section of $J^{\lambda}(TM)$.

To prove (4.6) we observe that if y is a point of T^rM and Z is a vector field on M such that $\sigma(\pi(y)) = j^{\lambda}_{\pi(y)}Z$, then by using (4.2), (3.7) and (4.1) we have

$$\begin{aligned} (F^H\sigma^{(\lambda)})(y) &= F^H_y(\sigma^{(\lambda)}(y)) \\ &= F^H_y(Z^{(\lambda)}(y)) \\ &= (FZ)^{(\lambda)}(y) \\ &= (F\sigma)^{(\lambda)}(y) \quad . \end{aligned}$$

The proof is now completed.

Now we shall prove the following theorem:

THEOREM 4.5. Let M be a manifold and $\Gamma^{(r)}$ be a connection of order r on M. If F is a complex structure on M such that

$$\overset{(r)}{\nabla}_X J^{r-1} FY = F(\overset{(r)}{\nabla}_X J^{r-1} Y) \tag{4.7}$$

$$R^{\square}(FX, FY) = R^{\square}(X, Y) \tag{4.8}$$

for all vector fields X and Y on M, then F^H is a complex structure on T^rM.

To prove this theorem we will need the following lemma.

LEMMA 4.6. If X is a vector field on M and σ is a section of $J^{\lambda}(TM)$, then for all $\eta \leq \lambda < r$ we have

$$\rho_{\eta}^{\lambda} \circ \overset{(\lambda+1)}{\nabla}_X \sigma = \overset{(\eta+1)}{\nabla}_X(\rho_{\eta}^{\lambda} \circ \sigma) .$$

PROOF. Let x be a point of M. Then

$$(\rho_{\eta}^{\lambda} \circ \overset{(\lambda+1)}{\nabla}_X \sigma)(x) = (\rho_{\eta}^{\lambda} \circ I_{\sigma(x)})((X^H \circ \sigma - d\sigma \circ X)(x))$$

$$= I_{\rho_{\eta}^{\lambda}(\sigma(x))}(d\rho_{\eta}^{\lambda}(X^H \circ \sigma - d\sigma \circ X)(x))$$

because $\rho_{\eta}^{\lambda} \circ I_{\sigma(x)} = I_{\rho_{\eta}^{\lambda}(\sigma(x))} \circ d\rho_{\eta}^{\lambda}$. We have also

$$d\rho_{\eta}^{\lambda} \circ X^H = X^H \circ \rho_{\eta}^{\lambda} ,$$

where X^H on the left hand side of the equality means the horizontal lift of X to $J^{\lambda}(TM)$ and X^H on the right hand side denotes the horizontal lift to $J^{\eta}(TM)$. Using the last formula we obtain

$$(\rho_{\eta}^{\lambda} \circ \overset{(\lambda+1)}{\nabla}_X \sigma)(x) = I_{(\rho_{\eta}^{\lambda} \circ \sigma)(x)}\left[(X^H \circ (\rho_{\eta}^{\lambda} \circ \sigma) - d(\rho_{\eta}^{\lambda} \circ \sigma) \circ X)(x)\right]$$

$$= (\overset{(\eta+1)}{\nabla}_X(\rho_{\eta}^{\lambda} \circ \sigma))(x) .$$

PROOF OF THEOREM 4.5. Since F is a complex structure, $N_F(X,Y) = 0$ for all vector fields X and Y on M. Hence by (4.8), (4.3) and (4.5) we get

$$N_{F^H}(X^{(\lambda)}, Y^{(\eta)}) = N_{F^H}(X^H, Y^H) = 0$$

for $\lambda, \eta = 0, \dots, r-1$. Next, using Lemma 4.6 and the formula (4.7) we have

$$\overset{(\lambda+1)}{\nabla}_X(J^{\lambda}FY) - F(\overset{(\lambda+1)}{\nabla}_X J^{\lambda}Y) = 0$$

for $\lambda = 0, \dots, r-1$, and hence, by using (4.4) we obtain

$$N_{F^H}(X^H, Y^{(\lambda)}) = 0 .$$

The equality $N_{F^H} = 0$ implies the integrability of F^H.

In the case of the tangent bundle $TM = T^1M$ Theorem 4.5 implies the results of K. Yano and S. Ishihara [12].

COROLLARY 4.7. (K. Yano, S. Ishihara [12]) Let M be a manifold and ∇ be a linear connection on M. If F is a complex structure on M such that

$$\nabla F = 0 \quad , \quad R(FX,FY) = R(X,Y)$$

for all vector fields X and Y on M, then F^H is a complex structure on TM.

Using the same argumentation as in Theorem 4.5 we can verify the following proposion:

PROPOSITION 4.8. Let $\Gamma^{(r)}$ be a connection of order r on a manifold M. If F is a product structure on M such that

$$\overset{(r)}{\nabla}_X(J^{r-1}FY) - F(\overset{(r)}{\nabla}_X J^{r-1}Y) = 0 \tag{4.9}$$

$$R^{\square}(FX,FY) + R^{\square}(X,Y) = 0 \tag{4.10}$$

for all vector fields X and Y on M, then F^H is a product structure on T^rM.

Since in the case of tangent bundle TM (r = 1) the equality (4.10) is equivalent to the following one

$$R(FX,FY) + R(X,Y) = 0 \quad , \tag{4.11}$$

we obtain:

COROLLARY 4.9. If ∇ is a linear connection on a manifold M and F is a product structure on M such that

$$\nabla F = 0 \quad , \quad R(FX,FY) + R(X,Y) = 0$$

for all vector fields X and Y, then F^H is a product structure on TM.

V. REMARK

In the same way we can define the horizontal lift of tensor fields of type (1.1) to the tangent bundle of p^r-velocities and we can obtain similar results.

REFERENCES

[1] CRITTENDEN R. "Covariant Differentiation", Quart. J. Math. Oxford, 13(1963), 285-296.

[2] GANCARZEWICZ J. "Connections of order r", Ann. Polon. Math. 34 (1977), 69-83.

[3] GANCARZEWICZ J. "Geodesics of order 2", Zeszyty Naukowe UJ, Prace Matematyczne 19(1977), 121-136.

[4] GANCARZEWICZ J. "Lifting of functions and vector fields to natura bundles", Diss. Math. CCXII, Warszawa 1983.

[5] GANCARZEWICZ J. and RAHMANI N. "Relèvements horizontaux des tenseurs de type (1.1) à $E = T^{*}M \otimes TM$", Ann. Pol. Math. (in the press).

[6] KOBAYASHI S. and NOMIZU K. "Foundations of differential geometry" vol. I, New York - London, 1963.

[7] de LEON M. and SALGADO M. "Diagonal Lifts of tensor fields to the frame bundle of second order", Rend. Circ. Mat. Palermo (in the press).

[8] MORIMOTO A."Prolongations of geometric structures", Lect. Notes Math. Inst. Nagoya Univ.,1967.

[9] MORIMOTO A. " Prolongations of G-stuctures to tangent bundles of higher order", Nagoya Math. J., 40(1970), 153-179.

[10] MORIMOTO A. "lifting of tensor fields and connections to tangent bundles of higher order", 40(1970), 99-120.

[11] RAHMANI N. "Relèvements horizintaux de tenseurs de type (1,1) à $E = T^p_qM$", Diff. Geometry,Proc. Conf. Nove Mesto 1983, University of Praha 1984, 117-126

[12] YANO K. and ISHIHARA S. "Horizontal lifts of tensor fields and connections to tangent bundles", J. Math. and Mech. 16(1967) 1015-1030.

[13] YANO K. and PATTERSON E. M. "Horizontal lifts from a manifold to its cotangent bundle", J. Math. Soc. Japan, 19(1967), 185-198.

[14] YUEN C. "Relèvement de dérivations aux fibrés tangent d'ordre 2", Comp. Rend. Ac. Sci. Paris, 282(1976), 703-706.

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