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M. Fabian and D. Preiss

Let $f: R^n \longrightarrow R^k$ be a locally Lipschitz function defined on an open ball $B(x,r) \subset R^n$ centered at x and of radius $r > 0$. According to the Rademacher's theorem [5] there exists a set $E_o \subset R^n$ of Lebesgue measure zero such that the Gâteaux derivative $Df(y)$ exists whenever $y \in B(x,r) \setminus E_o$. Using this fact Clarke [2] introduced the generalized Jacobian $\partial f(x)$ as the closed convex hull of all possible limits $\lim\limits_{y_i \to y} Df(y_i)$, where $y_i \in B(x,r) \setminus E_o$. Similarly, if E_o is replaced by a null set $E \subset R^n$ containing E_o, one can define $\partial_E f(x)$. Thus $\partial_{E_o} f(x) = \partial f(x)$. For $k = 1$ Clarke [1] showed that $\partial_E f(x) = \partial f(x)$ for any null set E containing E_o and asked in [2] if the equality remains true for $k > 1$. In what follows we answer this question affirmatively by showing

<u>Theorem.</u> $\partial_E f(x) = \partial f(x)$
for all k and for all null sets E including E_o.

<u>Proof.</u> All the spaces R^m are considered with the Euclidean norm $\|\cdot\|$. The symbol $\langle \cdot , \cdot \rangle$ denotes the usual inner product. The space of linear mappings from R^n to R^k as well as its dual will be identified with R^{nk}. Since clearly $\partial_E f(x) \subset \partial f(x)$, it remains to prove the converse. By contradiction, let us assume that this inclusion is proper. Then there exist a functional A in R^{nk} and $\alpha \in R$ such that

$$\sup \{\langle A,L \rangle : \ L \in \partial_E f(x)\} < \alpha < \sup \{\langle A,L \rangle : \ L \in \partial f(x)\} .$$

The definition of $\partial_E f(x)$ yields an $\varepsilon > 0$ such that

$$\langle A, Df(y) \rangle < \alpha \quad \text{whenever} \quad y \in B(x,\varepsilon) \setminus E .$$

Indeed, otherwise we could find $y_i \in B(x,1/i) \setminus E$ with

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$\langle A, Df(y_i)\rangle \geqq \alpha$ and hence there would exist an $L \in \partial_E f(x)$ such that $\langle A, L\rangle \geqq \alpha$, which is impossible. Also, according to the definition of $\partial f(x)$, there is $\bar{y} = (\bar{y}_1, \dots, \bar{y}_n) \in B(x,\varepsilon) \setminus E_0$ with

$$\alpha < \langle A, Df(\bar{y})\rangle .$$

By joining the last two inequalities we get

$$\langle A, Df(y)\rangle < \alpha < \langle A, Df(\bar{y})\rangle \quad \text{whenever} \quad y \in B(x,\varepsilon) \setminus E .$$

Denoting $A = (a_{ij})$, $i = 1,\dots,k$, $j = 1,\dots,n$, and $g_j = a_{1j}f_1 + \dots + a_{kj}f_k$, $j = 1,\dots,n$, we can write the above inequality in the form

$$\sum_{j=1}^{n} \frac{\partial g_j(y)}{\partial y_j} < \alpha < \sum_{j=1}^{n} \frac{\partial g_j(\bar{y})}{\partial y_j} \quad \text{whenever} \quad y \in B(x,\varepsilon) \setminus E .$$

Let $C(s)$ be the n-dimensional cube with apices $(\bar{y}_1 \pm s, \dots, \bar{y}_n \pm s)$. Whenever $s > 0$ is so small that $C(s) \subset B(x,\varepsilon)$, the above inequality holds almost everywhere in $C(s)$ and, consequently,

$$(*) \qquad \sum_{j=1}^{n} \int_{C(s)} \frac{\partial g_j(y)}{\partial y_j}\, dy_1 \dots dy_n < (2s)^n \alpha < (2s)^n \sum_{j=1}^{n} \frac{\partial g_j(\bar{y})}{\partial y_j} .$$

Let us denote

$$\delta_j(s) = \sup\left\{ |g_j(y) - g_j(\bar{y}) - Dg_j(\bar{y})(y-\bar{y})| : \max_{i=1,\dots,n} |y_i - \bar{y}_i| \leqq s \right\},$$

and

$$C_j(s) = \{(y_1,\dots,y_{j-1},y_{j+1},\dots,y_n): (y_1,\dots,y_{j-1},\bar{y}_j,y_{j+1},\dots,y_n) \in C(s)\}, \quad j = 1,\dots,n, \quad s > 0 .$$

Using the Fubini theorem, we get

$$\int_{C(s)} \frac{\partial g_j(y)}{\partial y_j}\, dy_1 \dots dy_n =$$

$$= \int_{C_j(s)} \sum_{\sigma = \pm 1} \sigma g_j(y_1,\dots,y_{j-1},\bar{y}_j + \sigma s, y_{j+1},\dots,y_n) \times$$

$$\times\, dy_1 \dots dy_{j-1} dy_{j+1} \dots dy_n =$$

$$= \int_{C_j(s)} \Bigg(\sum_{\sigma = \pm 1} \sigma \Big[g_j(y_1,\dots,y_{j-1},\bar{y}_j + \sigma s, y_{j+1},\dots,y_n) - g_j(\bar{y}) -$$

$$- Dg_j(\bar{y})(y_1 - \bar{y}_1,\dots,y_{j-1} - \bar{y}_{j-1}, \sigma s, y_{j+1} - \bar{y}_{j+1},\dots,y_n - \bar{y}_n)\Big] + 2s \frac{\partial g_j(\bar{y})}{\partial y_j}\Bigg) \times$$

$$\times\, dy_1 \dots dy_{j-1} dy_{j+1} \dots dy_n \geqq -2(2s)^{n-1}\delta_j(s) + 2s \frac{\partial g_j(\bar{y})}{\partial y_j} (2s)^{n-1} .$$

Hence $(*)$ implies

$$(2s)^n \sum_{j=1}^{n} \frac{\partial g_j(\bar{y})}{\partial y_j} - (2s)^n \sum_{j=1}^{n} \delta_j(s)/s < (2s)^n \alpha < (2s)^n \sum_{j=1}^{n} \frac{\partial g_j(\bar{y})}{\partial y_j}$$

if $s > 0$ is sufficiently small. Let us note that $\delta_j(s)/s \longrightarrow 0$ as $s \downarrow 0$ since Gâteaux and Fréchet differentiability in finite-dimensional spaces coincide for Lipschitz functions. Thus, dividing the above inequality by $(2s)^n$ and letting s go to zero, we obtain a wrong inequality. This contradiction finishes the proof.

Remark 1. For f and x as above Pourciau [4] considered a generalized Jacobian which in our notation is equal to $\partial_{E_1} f(x)$ with

$E_1 = E_0 \cup \{y \in B(x,r) \setminus E_0 : \ y$ is not a Lebesgue point of $Df\}$.

As f is locally Lipschitz, E_1 is a null set. Hence by Theorem $\partial_{E_1} f(x) = \partial f(x)$.

Remark 2. The reader probably noticed that the above proof is actually based on the Gauss - Green theorem. In fact, this theorem shows a "Denjoy property for derivatives of mappings between R^n and R^k" suggested by [2, Remark 5].

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