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CHOQUET SIMPLICES AND HARNACK INEQUALITIES

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Let S be a metrizable Choquet simplex, E = E(S) the set of all extreme points of S and $C(\overline{E})$ the vector space of all continuous functions on $\overline{E}$. Given a point $x \in S$, we introduce a further notation: $\mathcal{M}_x$ - the set of all lower semicontinuous affine functions

$s: S \to]-\infty, +\infty]$ with $s(x) < +\infty$,

μ_x - the (unique) maximal measure representing the point x,

face(x) - the smallest face of S containing x,

cl.face(x) - the smallest closed face of S containing x.

By the solution of the Dirichlet problem for a boundary function f on E we understand the affine function u_f defined on

$$D_f = \{x \in S: f \in L^1(\mu_x)\}$$

by

$$u_f(x) = \mu_x(f).$$

Notice that D_f is always a face.

Consider the heat equation on a relatively compact region $Q = \Omega \times T$, $\Omega \subset R^n$. As noticed in [1] , if we consider a compact set K and a point $x \in Q$ such that the time co-ordinate of any point of K is less than the time co-ordinate of x, then for each positive solution f of the heat equation on Q we have the Harnack inequality

$$\sup_{y \in K} f(y) \leq \alpha_K f(x).$$

The aim of this paper is to describe the points $x \in S$ for which an analogue of the Harnack inequality is satisfied on face(x): i.e. for any compact set $K \subset$ face(x) there is a number $\alpha_K \geq 1$ such that for every (continuous) affine function f: face(x) $\to [0, +\infty[$ we have

$$\sup_{y \in K} f(y) \leq \alpha_K f(x).$$

We prove that this is the case if and only if the restriction to face(x) of the solution of the Dirichlet problem is continuous for

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every boundary function from $L^1(\mu_x)$. Moreover, we show that if the considered face(x) possesses the additional property $\overline{\text{face}(x)} =$ $=$ cl.face(x), then the solution of the Dirichlet problem is continuous at all points of face(x) for every boundary function from $C(\overline{E})$. Notice that the property $\overline{\text{face}(x)}$ = cl.face(x) is natural for elliptic and parabolic equations.

Theorem 1. Let x be a point of S, $K \subset$ face(x) be a compact set and α be a number from $[1, +\infty[$. Then the following assertions are equivalent:

(i) for every continuous affine function $a: S \to [0, +\infty[$ we have

$$\sup_K a \leq \alpha\, a(x),$$

(ii) for every $\beta > \alpha$ we have

$$K \subset \beta x - (\beta - 1)\, S,$$

(iii) for every concave function s: face(x) $\to [0, +\infty]$ we have

$$\sup_K s \leq \alpha s(x).$$

Proof. (i)$\Rightarrow$(ii) cf. Theorem II.5.24 of [2] .

(ii)$\Rightarrow$(iii): Let $y \in K$ and $\beta > \alpha$. Then there is $z \in S$ with

$$y = \beta x - (\beta - 1) z.$$

Obviously $z \in$ face(x). Hence

$$\beta^{-1} s(y) \leq \beta^{-1}(\beta - 1) s(z) + \beta^{-1} s(y) \leq s(x).$$

Since $\beta > \alpha$ was arbitrary, the assertion follows.

(iii)$\Rightarrow$(i) is obvious.

Theorem 2. Let x be a point of S and $K \subset$ face(x) be a compact set. Then the following assertions are equivalent:

(i) (a Harnack type inequality) there is a constant $\alpha \in [1, +\infty[$ such that for every continuous affine function a: $S \to [0, +\infty[$ we have

$$\sup_K a \leq \alpha a(x),$$

(i^*) there is a constant $\alpha \in [1, +\infty[$ such that for every concave function s: face(x) $\to [0, +\infty]$ we have

$$\sup_K s \leq \alpha s(x),$$

(ii) (a Harnack type monotone convergence theorem) every increasing sequence $\{a_n\}$ of real affine functions on face(x) with $\sup a_n(x) < +\infty$ converges uniformly on K,

(iii) every increasing sequence $\{a_n\}$ of continuous real affine functions on S with $\sup_n a_n(x) < +\infty$ converges uniformly on K,

(iv) for every function $s \in \mathcal{M}_x$, the restriction $s|_K$ is continuous,

(v) for every function $s \in \mathcal{M}_x$, the restriction $s|_K$ is bounded,

(vi) for every function $f \in L^1(\mu_x)$, the restriction $u_f|_K$ is continuous.

Proof. (i)⇔(i*) by Theorem 1 .

(i)⇒(ii). We have

$$0 \leq a_{n+p} - a_n \leq \alpha(a_{n+p}(x) - a_n(x))$$

on K for every $n,p \in \mathbb{N}$. Hence the assertion follows.

(ii)⇒(iii) is obvious.

(iii)⇒(iv). According to Corollary I.1.4 of [2] and the separability of the space of all real continuous functions on S, there is an increasing sequence $\{a_n\}$ of real continuous affine functions on S with $\sup_n a_n = s$. By (iii), $\{a_n\}$ converges uniformly to s on K and hence $s|_K$ is continuous.

(v)⇒(i). Assume that for every $n \in \mathbb{N}$ there is $x_n \in K$ and a continuous affine function $f_n : S \to]0,+\infty[$ such that

$$f_n(x_n) \geq n^3 f_n(x).$$

Consider the function

$$f = \sum_{k=1}^{\infty} \frac{f_k}{k^2 f_k(x)} .$$

Obviously $f \in \mathcal{M}_x$. We have

$$f(x_n) \geq \frac{f_n(x_n)}{n^2 f_n(x)} \geq n ,$$

which contradicts (v).

(iv)⇒(vi). We already know (iv)⇔(ii). Let g be a real continuous concave function on S. According to Theorem II.3.7 of [2] , $u_g \in \mathcal{M}_x$ and hence $u_g|_K$ is continuous by (iv). Since the set of all restrictions to $\overline{E}$ of differences of real continuous concave functions on S is dense in $C(\overline{E})$, we conclude $u_h|_K$ is continuous for every $h \in C(\overline{E})$. Using (ii) we deduce that $u_s|_K$ is continuous for every lower semicontinuous function $s: S \to]-\infty,+\infty]$ with $\mu_x(s) < +\infty$. Hence the functions

$$x \mapsto \int^* f \, d\mu_x, \quad x \mapsto \int^* (-f) \, d\mu_x , \quad x \in K$$

are upper semicontinuous, which yields the assertion.

(vi)⇒(iv) is obvious.

Corollary. Let $x \in S$. Then the following assertions are equivalent:
(i) for every compact set $K \subset \mathrm{face}(x)$ there is a constant $\alpha_K \in [1, +\infty[$ such that
$$\sup_K a \leqslant \alpha_K a(x)$$
for any continuous affine function $a: \mathrm{face}(x) \to [0, \infty[$,
(ii) for every function $f \in L^1(\mu_x)$, the restriction $u_f \,|\, \mathrm{face}(x)$ is continuous.

Proof. Since $u_f \,|\, \mathrm{face}(x)$ is continuous if and only if $u_f|_K$ is continuous for every compact set $K \subset \mathrm{face}(x)$, the assertion follows from the preceding theorem.

Theorem 3. Let $x \in S$. Assume that one condition of the preceding corollary is satisfied and, moreover, $\mathrm{face}(x) = \mathrm{cl.face}(x)$. Then u_f is continuous at all points of $\mathrm{face}(x)$ for every $f \in C(\overline{E})$.

Proof. According to [3] is remains to verify that for every $y \in \mathrm{face}(x)$ the measure μ_y is the only probability measure representing y supported by $\overline{E}$. Assume that there is $y \in \mathrm{face}(x)$ and two different probability measures μ_y and ν_y on $\overline{E}$ representing y. Since $\mathrm{cl.face}(x)$ is a simplex and $\overline{\mathrm{face}(x)} = \mathrm{cl.face}(x)$, it follows from [4] that there are sequences $\{y_n\}$ and $\{x_n\}$ of points of $\mathrm{face}(x)$ with
$$\lim_n y_n = y, \quad \lim_n x_n = y$$
and
$$w - \lim \mu_{y_n} = \nu_y \ , \qquad w - \lim \mu_{x_n} = \mu_y \ .$$
Consider a function $f \in C(\overline{E})$ such that
$$\mu_y(f) \neq \nu_y(f).$$
Then
$$\lim_n u_f(y_n) \neq \lim_n u_f(x_n)$$
which contradicts the continuity of $u_f|_{\mathrm{face}(x)}$.

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