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THE LEFSCHETZ TYPE THEOREM FOR A CLASS OF NONCOMPACT MAPPINGS

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The purpose of this note is to present some new algebraic and topological notions related to the generalized trace theory of J. Leray and their connections with the fixed point theory. This is well known that the Leray trace plays a crucial role in the so-called Lefschetz theorem for compact mappings and some of their generalizations (see [1], [2], [3], [4]). The analogous results for other classes of mappings, e.g. A-proper mappings of Browder-Petryshyn [9], A-mappings [7], [5], F-mappings [7] and other mappings which arise naturally when studying the fixed point problems, are unknown yet. So, this is our aim to try to extend the algebraic tool of the Lefschetz theorem to these more general situations.

This is the first part of a larger research, and that is why we shall limit ourselves only to the sketch of an algebraic setting and its application to the class of A-mappings.

Moreover, we give theorems (see (8.5) and (8.6)) which seem to be interesting from the point of view of the asymptotic fixed point theory for compact mappings.

I. Trace theory

In spite of the fact that we shall need only some of the forthcoming results, we present them (in the sketchy form), for the sake of completeness, together with some others. It seems that this theory may be of interest of its own.

1. Let us recall some fundamental notions. For a finite-dimensional vector space (VS) F over a field K we define two homomorphisms $\theta : F \otimes_K F \to \mathrm{End}\,(F)$ and $e : F \otimes_K F \to K$ given (on generators) by the formulae $\theta(f \otimes x)(x') = f(x')x$ and $e(f \otimes x) = f(x)$. It is quite easy to see that θ is an isomorphism. We define the <u>ordinary trace</u> of an endomorphism $\varphi \in \mathrm{End}(F)$ by: $\mathrm{tr}\,\varphi = e(\theta^{-1}\varphi)$.

Here are the most useful properties of tr.

(1.1) (i) <u>Let the following diagram of finite-dimensional</u> VS's

This paper is in final form and no version of it will be submitted for publication elsewhere.

over K and homomorphisms commute

$$\begin{tikzcd} F \arrow[r, "f"] \arrow[d, "\varphi"'] & F' \arrow[d, "\varphi'"] \arrow[dl, "g"] \\ F \arrow[r, "f"'] & F', \end{tikzcd}$$

then $\operatorname{tr} \varphi = \operatorname{tr} \varphi'$.

(ii) If the following diagram of finite-dimensional VS's and homomorphisms commutes and has exact rows

$$\begin{tikzcd} 0 \arrow[r] & F' \arrow[r] \arrow[d, "\varphi'"] & F \arrow[r] \arrow[d, "\varphi"] & F'' \arrow[r] \arrow[d, "\varphi''"] & 0 \\ 0 \arrow[r] & F' \arrow[r] & F \arrow[r] & F'' \arrow[r] & 0, \end{tikzcd}$$

then $\operatorname{tr} \varphi = \operatorname{tr} \varphi' + \operatorname{tr} \varphi''$.

2. Now, let F be an arbitrary VS over K and let $\varphi \in \operatorname{End}(F)$. We put $N\varphi = \bigcup_{n \geq 1} \ker \varphi^n$. It is easily seen that $\varphi^{-1}(N\varphi) = N\varphi$, hence φ induces a monomorphism $\tilde{\varphi} : \tilde{F} \to \tilde{F}$ where $\tilde{F} = F/N\varphi$. We say that φ is a Leray endomorphism (an L-endomorphism) if $\dim_K \tilde{F} < \infty$ and we define the Leray trace $\operatorname{Tr} \varphi$ of φ by setting $\operatorname{Tr} \varphi = \operatorname{tr} \tilde{\varphi}$. Observe by (1.1) (ii), that if F is finite-dimensional and $\varphi \in \operatorname{End}(F)$, then φ is an L-endomorphism and $\operatorname{Tr} \varphi = \operatorname{tr} \varphi$.

Next (see [8]).

(2.1) (i) If the diagram of VS's and homomorphisms

$$\begin{tikzcd} F \arrow[r, "f"] \arrow[d, "\varphi"'] & F' \arrow[d, "\varphi'"] \arrow[dl, "g"] \\ F \arrow[r, "f"'] & F' \end{tikzcd}$$

commutes and φ is an L-endomorphism, then φ' is such and $\operatorname{Tr} \varphi = \operatorname{Tr} \varphi'$.

(ii) If the diagram of VS's and homomorphisms

$$\begin{tikzcd} 0 \arrow[r] & F' \arrow[r] \arrow[d, "\varphi'"] & F \arrow[r] \arrow[d, "\varphi"] & F'' \arrow[r] \arrow[d, "\varphi''"] & 0 \\ 0 \arrow[r, dash] & F \arrow[r] & F \arrow[r] & F'' \arrow[r] & 0 \end{tikzcd}$$

commutes and has exact rows and φ is an L-endomorphism, or, φ', φ'' are L-endomorphisms, then $\varphi, \varphi', \varphi''$ are L-endomorphisms and $\operatorname{Tr} \varphi =$

$= \mathrm{Tr}\ \varphi' + \mathrm{Tr}\ \varphi''$.

As an easy consequence we get the following fact.

(2.2) *Let F be a VS over K and $\varphi \in \mathrm{End}(F)$. If there exists a finite-dimensional vector subspace F' of F such that $\varphi(F') \subset F'$ and, for each $x \in F$, there is $n = n(x)$ such that $\varphi^n(x) \in F'$, then φ is an L-endomorphism and $\mathrm{Tr}\ \varphi = \mathrm{tr}\ (\varphi|F')$.*

P r o o f. The following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F' & \longrightarrow & F & \longrightarrow & F/F' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle (\varphi|F')} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi''} & & \\ 0 & \longrightarrow & F' & \longrightarrow & F & \longrightarrow & F/F' & \longrightarrow & 0, \end{array}$$

where φ'' is induced by φ, is commutative and has exact rows. Since F' is finite-dimensional, $\varphi|F'$ is an L-endomorphism and $\mathrm{Tr}\ (\varphi|F') = \mathrm{tr}\ (\varphi|F')$. Next, $N\varphi'' = F/F'$. So, φ'' is an L-endomorphism, too, and $\mathrm{Tr}\ \varphi'' = 0$. By (2.1)(ii), we end the proof. q.e.d.

(2.3) *If $\varphi \in \mathrm{End}\ (F)$ and $\dim \mathrm{Im}\ \varphi^n < \infty$ for some $n \geq 1$, then φ is an L-endomorphism.*

3. Although very general, the above theory does not cover many natural situations.

(3.1) *Example*. Let F be a VS over K generated by the set Z of integers, i.e. the space of all functions $Z \to K$ with finite supports. Let $\alpha : Z \to K$ be a function such that $\alpha(x) \neq 0$ for all $x \in Z$. We define an endomorphism $\varphi : F \to F$ by the formula $\varphi(\sum_{i=1}^{p} a_i x_i) = \sum_{i=1}^{p} a_i \alpha(x_i) x_i$ where $x_i \in Z$, $a_i \in K$, $i = 1,2,\dots,n$. We see that φ is a monomorphism, hence $N\varphi = \{0\}$ and $\dim_K \tilde{F} = \infty$. Thus φ is not an L-endomorphism. But formally, one can treat the series $\sum_{x \in Z} \alpha(x)$ (even if not convergent), i.e. the family $\{\sum_{x \in T} \alpha(x)\}_{T \subset Z}$, $\mathrm{card}\ T < \infty$, as a generalization of the notion of the trace.

Below, we shall construct a theory which makes it possible to deal with situations similar to that described above.

Let $(S, \leq)$ be a *directed set* and let $\varepsilon = \{E_s, i_{st} : E_s \to E_t\}_{s \in S}$ be a *direct system* of VS's (over K). We say that a pair $(F, \{f_s : E_s \to F\}_{s \in S})$, where F is a VS and f_s is a homomorphism for any $s \in S$, is *compatible with* ε if the diagram

$$\begin{tikzcd} E_s \arrow[d, "i_{st}"'] \arrow[dr, "f_s"] & \\ E_t \arrow[r, "f_t"'] & F \end{tikzcd}$$

is commutative for $s,t \in S$, $s \leq t$. Let pairs $(F,\{f_s\})$, $(F',\{f'_s\})$ be compatible with ε. We say that $f : F \to F'$ is a *homomorphism of these pairs* if the diagram

$$\begin{tikzcd} & F \arrow[d, "f"] \\ E_s \arrow[ur, "f_s"] \arrow[r, "f'_s"'] & F' \end{tikzcd}$$

is commutative for each $s \in S$. We write $f : (F,\{f_s\}) \to (F',\{f'_s\})$.

Having a direct system $\varepsilon = \{E_s, i_{st}\}_{s\in S}$ one can construct (see [10]) the compatible pair $(E,\{i_s\})$ called the *direct limit of* ε and denoted by $\lim\limits_{s\in S} \varepsilon$.

(3.1) [10]. *The following properties are satisfied*

(i) $\bigcup\limits_{s\in S} i_s(E_s) = E$

(ii) *For each* $s \in S$, $\ker i_s = \bigcup\limits_{t\geq s} \ker i_{st}$.

(iii) $\lim\limits_{s\in S} \varepsilon$ *is characterized up to isomorphism of pairs by the property that, given a compatible pair* $(F,\{f_s\})$, *there is a unique homomorphism of pairs* $f : \lim\limits_{s\in S} \varepsilon \to (F,\{f_s\})$ *which will be denoted by* $(f_s)_{s\in S}$.

As a consequence one has the following simple corollary.

(3.2) *If a pair* $(F,\{f_s\})$ *is compatible with* ε, *then* $(f_s)_{s\in S}$ *is an isomorphism if and only if the following conditions are satisfied*:

(i) $\bigcup\limits_{s\in S} f_s(E_s) = F$,

(ii) $f_s(x_s) = f_t(x_t)$ for $s,t \in S$, $x_s \in E_s$, $x_t \in E_t$ iff there is $u \geq s,t$ such that $i_{su}(x_s) = i_{tu}(x_t)$.

Suppose F is a VS over K, $\varphi \in \mathrm{End}\,(F)$ and $\varepsilon = \{E_s, i_{st}\}_{s\in S}$ is a direct system of VS's. If there exist a cofinal subset $S' \subset S$, a family $\{f_s : E_s \to F\}_{s\in S'}$ such that a pair $(F,\{f_s\}_{s\in S'})$ is compatible with $\varepsilon' = \{E_s, i_{st}\}_{s\in S'}$ and $(f_s)_{s\in S'} : \lim\limits_{s\in S'} \varepsilon' \to (F,\{f_s\}_{s\in S'})$ is an isomorphism, a morphism of direct systems (see [10]), $\{\varphi_s\}_{s\in S'}: \varepsilon' \to \varepsilon'$ such that the diagram

$$\begin{array}{ccc} E_s & \xrightarrow{f_s} & F \\ {\scriptstyle \varphi_s}\downarrow & & \downarrow{\scriptstyle \varphi} \\ E_s & \xrightarrow{f_s} & F \end{array}$$

commutes for all $s \in S'$, then we say that φ is <u>decomposable with respect to</u> (w.r.t.) ε and call the triple $\mathcal{D} = (\varepsilon', \{f_s\}_{s\in S'}, \{\varphi_s\}_{s\in S'})$ <u>a decomposition of</u> φ w.r.t. ε. If there exists a direct system ε such that φ is decomposable w.r.t. ε, then we say that φ is <u>decomposable</u>.

A decomposition $\mathcal{D}$ is called <u>injective</u> if f_s is a monomorphism for $s \in S'$. It is easily seen, by (3.1) (i), (ii), that it is equivalent to the injectivity of i_s for any $s \in S'$.

Obviously, any endomorphism $\varphi : F \to F$ has a decomposition namely, the trivial one, i.e. $E_s = F$, $f_s = id_F$ and $\varphi_s = \varphi$ for every $s \in S$.

(3.3) <u>If an endomorphism</u> $\varphi \in End(F)$ <u>has a (nontrivial) decomposition, then it has an injective decomposition, as well.</u>

(3.4) <u>For</u> $\varphi \in End(F)$ <u>to have a nontrivial decomposition it is necessary and sufficient that there exist a directed set</u> S <u>and an increasing family</u> $\{F_s\}_{s\in S}$, <u>i.e.</u> $F_s \subset F_t$ <u>for</u> $s \le t$, <u>of nontrivial vector subspaces of</u> F <u>such that</u> $\bigcup_{s\in S} F_s = F$ <u>and</u> $\varphi(F_s) \subset F_s$ <u>for</u> $s \in S$.

Now, let $\varphi \in End(F)$ and let $\varepsilon = \{E_s, i_{st}\}_{s\in S}$ be a direct system of VS's. Assume $\mathcal{D} = (\varepsilon', \{f_s\}_{s\in S'}, \{\varphi_s\}_{s\in S'})$ to be a decomposition of φ w.r.t. ε. We shall say that φ is <u>L-decomposable w.r.t.</u> ε, $\mathcal{D}$ is an <u>L-decomposition of w.r.t.</u> ε and φ is a <u>generalized Leray endomorphism</u> (generalized L-endomorphism) if there is $s_o \in S'$ such that, for $s \in S'$, $s \ge s_o$, φ_s is an L-endomorphism.

In the set $\prod_{s\in S} K_s$, where $K_s = K$ for any $s \in S$, we introduce an equivalence relation "~" defined as follows: $(a_s)_{s\in S} \sim (b_s)_{s\in S}$ iff there is $s_o \in S$ such that $a_s = b_s$ for $s \ge s_o$. The equivalence class of $(a_s)_{s\in S} \in \prod_{s\in S} K_s$ is denoted by $[(a_s)_{s\in S}]$.

For a generalized L-endomorphism φ with an L-decomposition $\mathcal{D} = (\varepsilon', \{f_s\}_{s\in S'}, \{\varphi_s\}_{s\in S'})$ such that, for $s \in S'$, $s \ge s_o \in S'$,

φ_s is an L-endomorphism, we define

$$a_s = \begin{cases} \mathrm{Tr}\ \varphi_s & \text{for } s \in S',\ s \geq s_o \\ 0 & \text{for other } s. \end{cases}$$

Next, we define the <u>generalized Leray trace of</u> φ <u>w.r.t.</u> as an element of $\prod_{s\in S} K_s/\sim$ given by

$$\mathrm{Tr}\ (\varphi,\mathcal{D}) = [(a_s)_{s\in S}].$$

As is easily seen, the endomorphism φ from (3.1) is a generalized L-endomorphism w.r.t. a direct system $\varepsilon = \{E_T, i_{TU} : E_T \to E_u\}$ where $T \subset U \subset Z$, card $U < \infty$, E_T is the vector subspace of F generated by T. If $\mathcal{D} = (\varepsilon, \{f_T\}, \{\varphi_T\})$ is a decomposition of φ w.r.t. ε, then $\mathrm{Tr}\ (\varphi,\mathcal{D}) = [(\sum_{x\in T} \alpha(x))_{T\subset Z, \mathrm{card}\ T<\infty}]$.

It seems to be obvious that the notion of the generalized trace depends strongly on the choice of a decomposition and a direct system.

(3.5) <u>Example</u>. Let F be as in (3.1). Let $\varepsilon = \{E_T, i_{TU}\}$ and $\bar{\varepsilon} = \{\bar{E}_T, \bar{i}_{TU}\}$ where $\bar{E}_T = E_{-T}$. We take $f_T : E_T \hookrightarrow F$ and $\bar{f}_T = f_{-T}$, $\varphi_T = \varphi|E_T$, $\bar{\varphi}_T = \varphi|\bar{E}_T$. Then we have two distinct L-decompositions $\mathcal{D}$ and $\bar{\mathcal{D}}$ of φ w.r.t. ε and $\bar{\varepsilon}$, respectively, for which $\mathrm{Tr}(\varphi,\mathcal{D}) \neq$ $\neq \mathrm{Tr}\ (\varphi,\bar{\mathcal{D}})$.

However, the following simple proposition holds.

(3.6) <u>Let</u> $\varphi \in \mathrm{End}\ (F)$, <u>and let</u> $\varepsilon = \{E_s, i_{st}\}_{s\in S}$, $\bar{\varepsilon} = \{\bar{E}_s, i_{st}\}_{s\in S}$ <u>be two direct systems of</u> VS's. <u>Let</u> $\mathcal{D} = (\varepsilon', \{f_s\}_{s\in S'}, \{\varphi_s\}_{s\in S'})$ <u>be an injective</u> L-<u>decomposition of</u> φ w.r.t. ε <u>and let</u> $\bar{\mathcal{D}} = (\bar{\varepsilon}', \{\bar{f}_s\}_{s\in S'}, \{\bar{\varphi}_s\}_{s\in S'})$ <u>be an injective decomposition of</u> φ w.r.t. $\bar{\varepsilon}$. <u>If there exists a morphism</u> $\{\psi_s\}_{s\in S'} : \varepsilon' \to \bar{\varepsilon}'$, <u>with</u> ψ_s <u>being an isomorphism for any</u> $s \in S'$, <u>such that the diagram</u>

$$\begin{tikzcd} F \arrow[r, "f'\cdot\lim\{\psi_s\}\cdot f^{-1}"] \arrow[d, "\varphi"'] & F \arrow[d, "\varphi"] \\ F \arrow[r, "f'\cdot\lim\{\psi_s\}\cdot f^{-1}"'] & F, \end{tikzcd}$$

<u>where</u> $f = (f_s)_{s\in S'}$ <u>and</u> $f' = (f_s)_{s\in S'}$, <u>is commutative, then</u> $\bar{\mathcal{D}}$ <u>is an</u> L-<u>decomposition and</u> $\mathrm{Tr}\ (\varphi,\mathcal{D}) = \mathrm{Tr}\ (\varphi,\bar{\mathcal{D}})$.

The proof is simple and requires some technical, algebraic construction, so we shall omit it here.

For an L-endomorphism $\varphi \in \mathrm{End}\,(F)$, the trivial decomposition is an L-decomposition, but also

(3.7) *Any injective decomposition* $\mathcal{D}$ *of* φ *is an L-decomposition and* $\mathrm{Tr}\,(\varphi,\mathcal{D}) = [(\mathrm{Tr}\,\varphi)]$ (*the class of the constant family*).

The partial converse of (3.7) is given in

(3.8) *If* $\varphi \in \mathrm{End}\,(F)$ *has an injective L-decomposition* $\mathcal{D}$, *then* $\tilde{\varphi}$ *has such a decomposition* $\tilde{\mathcal{D}}$, *being finite-dimensional, and* $\mathrm{Tr}\,(\varphi,\mathcal{D}) = \mathrm{Tr}\,(\tilde{\varphi},\tilde{\mathcal{D}})$.

(3.9) *Let* $\varphi \in \mathrm{End}\,(F)$ *have a decomposition* $\mathcal{D} = (E',\{f_s\}_{s\in S'},\{\varphi_s\}_{s\in S'})$ *w.r.t. a direct system* ε. *If there exists a vector subspace* $F' \subset F$ *such that* $\varphi(F') \subset F'$ *and, for each* $x \in F$, *there is* $n = n(x)$ *such that* $\varphi^n(x) \in F'$, *and for* $s \in S'$, $s \geq s_o \in S'$, $\dim_K f_s^{-1}(F') < \infty$, *then* $\mathcal{D}$ *is an L-decomposition*.

To prove this it is sufficient to observe that $\varphi_s(f_s^{-1}(F')) \subset f_s^{-1}(F')$ and, for each $x \in E_s$, $\varphi_s^n(x) \in f_s^{-1}(F')$ where $n = n(f_s(x))$, then recall (2.2) for $s \in S'$, $s \geq s_o$.

Now, we present results analogous to (2.1).

(3.10) (i) *Let the following diagram of VS's and homomorphisms*

$$\begin{tikzcd} F \arrow[r, "\xi"] \arrow[d, "\varphi"'] & F' \arrow[d, "\varphi'"] \arrow[dl, "\psi"'] \\ F \arrow[r, "\xi"'] & F' \end{tikzcd}$$

be commutative. If φ *has an injective L-decomposition* $\mathcal{D}$, *then* φ' *has an injective L-decomposition* $\mathcal{D}'$, *and* $\mathrm{Tr}\,(\varphi,\mathcal{D}) = \mathrm{Tr}\,(\varphi',\mathcal{D}')$.

(ii) *If the diagram of VS's and homomorphisms*

$$\begin{tikzcd} 0 \arrow[r] & F' \arrow[r, "w'"] \arrow[d, "\varphi'"] & F \arrow[r, "p''"] \arrow[d, "\varphi''"] & F'' \arrow[r] \arrow[d, "\varphi''"] & 0 \\ 0 \arrow[r] & F' \arrow[r, "w'"] & F \arrow[r, "p''"] & F'' \arrow[r] & 0 \end{tikzcd}$$

is commutative, has exact rows and φ *has an injective L-decomposition* $\mathcal{D}$, *then* φ',φ'' *have injective L-decomposition* $\mathcal{D}'$ *and* $\mathcal{D}''$, *respectively, for which*

$$(*) \qquad \mathrm{Tr}\,(\varphi,\mathcal{D}) = \mathrm{Tr}\,(\phi',\mathcal{D}') + \mathrm{Tr}\,(\varphi'',\mathcal{D}'').$$

If there exists a projection $p : F \to F$ such that $\ker p = \ker p''$ and $\varphi p = p\varphi$, φ' and φ'' have injective L-decompositions, then φ has an injective L-decomposition $\mathcal{D}$ such that (*) holds.

Proof. We shall prove (i). The proof of (ii) runs similarly. Let $\mathcal{D} = (\{E_s, i_{st}\}_{s\in S'}, \{f_s\}_{s\in S'}, \{\varphi_s\}_{s\in S'})$ be an injective L-decomposition of φ w.r.t. a direct system $\varepsilon = \{E_s, i_{st}\}_{s\in S}$. Consider the following diagram

$$\begin{array}{ccc} F/\ker & \xrightarrow{\bar{\xi}} & \operatorname{Im}\xi \\ {\scriptstyle\bar{\varphi}}\downarrow & & \downarrow{\scriptstyle\varphi'} \\ F/\ker & \xrightarrow[\bar{\xi}]{} & \operatorname{Im}\xi \end{array}$$

where $\bar{\varphi}, \bar{\xi}$ are induced homomorphisms. It is commutative. Let $A_s = f_s^{-1}(\ker \xi)$. It is easy to verify that $\bar{\varepsilon} = \{E_s/A_s, \bar{i}_{st} : E_s/A_s \to E_t/A_t\}_{s\in S'}$ is a direct system and a pair $(F/\ker \xi, \{\bar{f}_s : E_s/A_s \to F/\ker)_{s\in S'}$ is compatible with $\bar{\varepsilon}$. By (3.2), $(\bar{f}_s)_{s\in S'}$ is an isomorphism. Thus $\bar{\mathcal{D}} = (\bar{\varepsilon}, \{\bar{f}_s\}, \{\bar{\varphi}_s\})$, where $\bar{\varphi}_s : E_s/A_s \to E_s/A_s$ for $s \in S'$, is induced by φ_s and is an injective L-decomposition of $\bar{\varphi}$, in view of (2.1)(ii). Since $\bar{\xi}$ is an isomorphism, we gather that $\mathcal{D}'' = (\bar{\varepsilon}, \{\bar{\xi} \cdot f_s\}, \{\bar{\varphi}_s\})$ is an L-decomposition of $\varphi'|\operatorname{Im}\xi$. Now, since $\varphi'(F') \subset \operatorname{Im}\xi$, we construct an L-decomposition $\mathcal{D}'$ of φ' such that $\operatorname{Tr}(\varphi', \mathcal{D}') = \operatorname{Tr}(\varphi'|\operatorname{Im}\xi, \mathcal{D}'') = \operatorname{Tr}(\bar{\varphi}, \bar{\mathcal{D}})$. Consider the following well-defined diagram

$$\begin{array}{ccccccc} E_s/A_s & \xrightarrow{\bar{\xi}\cdot\bar{f}_s} & \operatorname{Im}\xi\cdot f_s & \xleftarrow{\xi} & \operatorname{Im} f_s & \xleftarrow{f_s} & E_s \\ {\scriptstyle\bar{\varphi}_s}\downarrow & & \downarrow{\scriptstyle\varphi'} & \searrow & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi_s} \\ E_s/A_s & \xrightarrow[\bar{\xi}\cdot f_s]{} & \operatorname{Im}\xi\cdot f_s & \xleftarrow[\xi]{} & \operatorname{Im} f_s & \xleftarrow[f_s]{} & E_s \end{array}$$

which is commutative for each $s \in S'$. Our assertion now follows from (2.1) (i). q.e.d.

4. Let $F = \{F_q\}_{q\geq 0}$ be a graded VS over K. We say that F is of finite type if $\dim_K F_q < \infty$ for any $q \geq 0$ and $F_q = \{0\}$ for almost all q. If $\varphi = \{\varphi_q\}_{q\geq 0}$ is an endomorphism of F of degree 0, then we define the (ordinary) Lefschetz number $\lambda(\varphi)$ of φ by the formula:

$$\lambda(\varphi) = \sum_{q\geq 0} (-1)^q \operatorname{tr} \varphi_q .$$

Suppose now that $F = \{F_q\}_{q\geq 0}$ is an arbitrary graded VS and

$\varphi = \{\varphi_q\}_{q\geq 0}$ is an endomorphism (of degree 0) of F. We say that φ is a Leray endomorphism (L-endomorphism) if $\tilde{F} = \{\tilde{F}_q\}_{q\geq 0}$ is of finite type and, in this case, we define the Lefschetz number $\Lambda(\varphi)$ by the formula

$$\Lambda(\varphi) = \sum_{q\geq 0} (-1)^q \operatorname{Tr} \varphi_q .$$

It is obvious that if F is of finite type, then $\Lambda(\varphi) = \lambda(\varphi)$.

Now, we extend the above notions to a larger class of graded VS's. Let $F = \{F_q\}_{q\geq 0}$ be a graded VS and let $\varphi = \{\varphi_q\}_{q\geq 0}$ be an endomorphism of F. Suppose that, for each $q \geq 0$, $\varepsilon_q = \{E_{qs}, i_{qst}\}_{s\in S}$ is a direct system of VS's and $\mathcal{D}_q = (\varepsilon'_q, \{f_{qs}\}_{s\in S'}, \{\varphi_{qs}\}_{s\in S'})$ is a decomposition of φ_q w.r.t. ε_q. Let $\mathcal{D} = \{\mathcal{D}_q\}_{q\geq 0}$. We say that φ is L-decomposable w.r.t. $\varepsilon = \{\varepsilon'_q\}_{q\geq 0}$, φ is a generalized Leray endomorphism (generalized L-endomorphism) and $\mathcal{D}$ is an L-decomposition w.r.t. ε if, for any $s \in S'$, $s \geq s_o \in S'$, $\{\varphi_{qs}\}_{q\geq 0}$ is an L-endomorphism of the graded VS $\{E_{qs}\}_{q\geq 0}$. In this case, we define the generalized Lefschetz number of φ w.r.t. $\mathcal{D}$ by putting

$$\Lambda(\varphi, \mathcal{D}) = \sum_{q\geq 0} (-1)^q \operatorname{Tr} (\varphi_q, \mathcal{D}_q).$$

Now, which is important, one can easily restate the results of sections 1, 2, 3 to get the analogous properties of $\lambda(\varphi), \Lambda(\varphi)$ and $\Lambda(\varphi, \mathcal{D})$.

II. Uniform spaces and filtrations

We shall now apply the algebraic theory developed above to the fixed point theory of a certain type of (noncompact) mappings acting in uniform spaces.

In all what follows, by space we shall understand a Hausdorff uniform space, by mapping a continuous transformation. If X is a space with the uniform structure $\mathcal{X}$, then by vicinity (of the diagonal in $X \times X$) we mean an arbitrary $V \in \mathcal{X}$ open (in the product topology of $X \times X$), if E is a locally convex topological vector space (LCTVS), then by neighbourhood (nghbd) we mean a neighbourhood of the origin o in E. On subsets of a space we shall always consider the induced topology (and the uniform structure) of a subspace.

5. First, we shall recall and introduce some concepts and notations which are necessary in the sequel.

Let $(X,\mathcal{X})$ be a space and $Z \subset X$. For $V \in \mathcal{X}$ we put $V(Z) = \{y \in X \mid (z,y) \in V$ for some $z \in Z\}$; if V is a nghbd in an LCTVS E, then, for $Z \subset E$, $V(Z) = V + Z$. Let Y be a space and $U \in \mathcal{X}$. Two mappings $f,g : Y \to X$ are said to be U-homotopic, provided there is a mapping $h : Y \times [0,1] \to X$ such that $h(\cdot,0) = f$, $h(\cdot,1) = g$ and, for each $y \in Y$, there is $x \in X$ such that $h(y,t) \in U(x)$ for all $t \in [0,1]$.

6. Let X be a space. By a filtration we understand a family $\{X_s\}_{s\in S}$ where S is a directed set, such that $X_s \subset X_t$ if $s \le t$, and $\mathrm{cl}\,(\bigcup_{s\in S} X_s) = X$. By $i_s : X_s \to X$ we denote the identity embedding. In particular, if X is an LCTVS and, for each $s \in S$, X_s is a linear subspace of X, then the filtration $\{X_s\}_{s\in S}$ is called a linear filtration.

We shall give some examples. Since any uniform space may be uniformly embedded in an LCTVS (this simple statement follows easily, as a corollary, from the well-known theorem due to Kuratowski) and linear filtrations play a crucial role in the sequel, subsets of an LCTVS create the most important examples.

(6.1) Example. (i) Let X be a space and Y an open subset of X. If $\{X_s\}_{s\in S}$ is a filtration in X, then $\{Y_s\}_{s\in S}$, where $Y_s = Y \cap X_s$, is a filtration in Y.

(ii) If $Y \subset X$, where X is a space, is filtrated by $\{Y_s\}_{s\in S}$, then $\mathrm{cl}\, Y$ is also filtrated by $\{Y_s\}$ and by $\{\mathrm{cl}\, Y_s\}$.

(iii) Let G be an open, convex nghbd in an LCTVS E, $G \ne E$, and let $\{E_s\}_{s\in S}$ be an increasing family of vector subspaces such that $\mathrm{cl}\,(\bigcup_{s\in S} E_s) = E$. If $B = \mathrm{bd}\, G$ is the boundary of G, then $\{B_s\}_{s\in S}$, where $B_s = B \cap E_s$, is a filtration in B.

Only the last part needs a proof. Take $x \in B$ and an arbitrary convex nghbd V. By (i), $\{(E \setminus \mathrm{cl}\, G) \cap E_s\}_{s\in S}$, $\{G \cap E_s\}$ are filtrations in $E \setminus \mathrm{cl}\, G$ and G, respectively. Hence there are points $y' \in V(x) \cap (E \setminus \mathrm{cl}\, G) \cap E_s$ and $y'' \in V(x) \cap G \cap E_s$ for sufficiently large s. We denote by p the Minkowski gauge of G. Since $p(y') > 1$ and $p(y'') < 1$, there must be a point y lying on the segment joining y' and y'', thus belonging to $V(x) \cap E_s$, such that $p(y) = 1$ and, hence, $y \in B$.

The notion of a filtration is not sufficient for our purposes. We shall need a more complex object.

Let $(X,\mathcal{X})$ be a space. We say that a filtration $\{X_s\}_{s\in S}$ of X is regular over a subset $Z \subset X$, if, for each $U \in \mathcal{X}$, there are $V \in \mathcal{X}$ and $s_0 \in S$ such that, for any $s \geq s_0$, there exists a mapping $\pi_s : V(X_s) \cap V(Z) \to X_s$ such that $\pi_s(x) = x$ for $x \in X_s \cap V(Z)$ and $i_s \circ \pi_s$, $i : V(X_s) \cap V(Z) \to X$ are U-homotopic. We shall say that $\{X_s\}$ is regular if it is regular over the entire space X.

Recall that a topological space Y is said to be r-dominated by a space G if there are mappings $r : G \to Y$ and $j : Y \to G$ such that $r \circ j : Y \to Y$ is the identity mapping id_Y.

We say that a filtration $\{X_s\}_{s\in S}$ of a space $(X,\mathcal{X})$ satisfies the condition (R) over $Z \subset X$ if:

(R) There are $T \in \mathcal{X}$ and $s_1 \in S$ such that, for each $s \geq s_1$, $T(Z) \cap X_s$ is r-dominated by an open subset of a convex set lying in an LCTVS. (In other words, see [3], we demand that $T(Z) \cap X_s$ be a Borsuk space).

(6.2) Example. Let E be a metrizable LCTVS filtrated by an increasing family $\{E_s\}_{s\in S}$ of finite-dimensional vector subspaces of E.

(i) If X is an open subset of E and, for $Z \subset X$, there is a nghbd W such that $W(Z) \subset X$, then a filtration $\{X_s = X \cap E_s\}_{s\in S}$ of X is regular and satisfies (R) over Z. Hence $\{X_s\}_{s\in S}$ is regular and satisfies the condition (R) over any compact subset of X.

(ii) If C is a convex subset of E filtrated by $\{C_s = C \cap E_s\}_{s\in S}$, then $\{C_s\}$ is regular and satisfies (R).

(iii) Let G be as in (6.1). The filtration $\{B_s = B \cap E_s\}_{s\in S}$ is regular and satisfies (R).

(iv) Let X be an ANR (metric) with a trivial filtration $X_s = X$ for any $s \in S$. This filtration is regular and satisfies (R).

P r o o f. (i) Let d be a metric compatible with the topological and convex structure of E and let U be an arbitrary nghbd. Let $\varepsilon > 0$ be such that $\mathrm{cl}\, B(o,3\varepsilon) \subset U \cap W$ where $B(o,3\varepsilon) = \{y \in E \mid d(o,y) < 3\varepsilon\}$. Define $V = B(o,\varepsilon)$ and, for any $x \in V(X_s) \cap V(Z)$, let $d_x = d(x,E_s) \leq \varepsilon$. We define a multivalued mapping $\Phi : V(X_s) \cap V(Z) \to E_s$ for $s \in S$ by $\Phi(x) = \mathrm{cl}\, B(x,2d_x) \cap E_s$. Φ

has closed, convex and complete values. Moreover, Φ is lower semicontinuous. Indeed, for an open $D \subset E_s$, the set

$$\Phi^{+1}(D) = \{x \in E \mid \Phi(x) \cap D \neq \emptyset\} = \{x \in E \mid B(x,2d_x) \cap D \neq \emptyset\} =$$
$$= \{x \in E \mid x \in B(o,2d_x) + D\}$$

is open in E. Hence, by the Michael Selection Theorem, there is a mapping $\pi_s : V(X_s) \cap V(Z) \to E_s$ such that $\pi_s(x) \in \Phi(x)$. Obviously π_s satisfies the conditions of regularity. It is clear that, for any nghbd $T \subset W$, $T(Z) \cap X_s$ is a Borsuk space.

(ii) The proof is almost the same as in case (i).

(iii) Take any nghbd U and let $V = \frac{1}{2} U$. For any $s \in S$, we construct $\pi_s' : V(B_s) \to E_s$ as in (i). If p is the Minkowski gauge of G and $r(x) = x/p(x)$ for $x \notin p^{-1}(0)$, then $\pi_s = r \circ \pi_s'$: $V(B_s) \to B_s$ satisfies our conditions. Moreover, for any $s \in S$, B_s is an ANR.

Let H denote the <u>singular homology functor</u> with coefficients in a field K, from the category of topological spaces and continuous mappings to the category of graded VS's over K and homomorphisms of degree 0. Thus, for a space X, $H(X) = \{H_q(X)\}_{q\geq 0}$ where $H_q(X)$ is the q-th singular homology group of X, and, for a mapping $f : X \to Y$, $H(f) = \{H_q(f) : H_q(X) \to H_q(Y)\}$. We assume to be known that H satisfies all the Eilenberg-Steenrod axioms for homology.

Let X be a space with a filtration $\{X_s\}_{s\in S}$. By $i_{st} : X_s \to X_t$, $s \leq t$, $i_s : X_s \to X$ we denote the identity embeddings. It is easy to see that, for each $q \geq 0$, $\varepsilon_q = \{H_q(X_s), H_q(i_{st})\}_{s\in S}$ is a direct system of VS's and a pair $(H_q(X), \{H_q(i_s)\}_{s\in S})$ is compatible with ε_q.

We shall now prove a result which is essential for further considerations.

(6.3) <u>If a filtration</u> $\{X_s\}_{s\in S}$ <u>of</u> X <u>is regular over any compact subset of</u> X, <u>then</u>

$$\lim_{s\in S} \{H_q(X_s), H_q(i_{st})\} \cong H_q(X),$$

<u>and this isomorphism is realized by</u> $(H_q(i_s))_{s\in S}$.

P r o o f. According to (3.2) it is sufficient to prove that $\bigcup_{s\in S} H_q(i_s)(H_q(X_s)) = H_q(X)$ and that, for any q-homology classes

$c_s \in H_q(X_s)$, $c_t \in H_q(X_t)$, $H_q(i_s)(c_s) = H_q(i_t)(c_t)$ iff there is $u \geq s,t$ such that $H_q(i_{su})(c_s) = H_q(i_{tu})(c_t)$.

Let $c = [\tilde{c}] \in H_q(X)$ and let $\tilde{c} = \sum_{i=1}^{p} \alpha_i \sigma_i$, where $\alpha_i \in K$ and σ_i is a singular q-simplex for $i = 1,2,\dots,p$, be a q-cycle in X. By A we denote a support $\operatorname{supp} \tilde{c}$ of $\tilde{c}$, i.e. $A = \bigcup_{i=1}^{p} \sigma_i(\Delta_q)$ where Δ_q is the standard q-simplex in $\mathbb{R}^{q+1}$. Since $\{X_s\}_{s\in S}$ is regular over A, thus, for $U = X \times X$, there exist $V \in \mathcal{X}$ and $s_o \in S$ such that, for $s \geq s_o$, there is a mapping $\pi_s : V(X_s) \cap V(A) \to X_s$ for which $i_s \circ \pi_s$ is homotopic to $i : V(X_s) \cap V(A) \to X$. Since $\bigcup_{s\in S} X_s$ is dense in X, one can find $s_1 \geq s_o$ such that $A \subset V(X_{s_1})$. Let $g = i_{s_1} \circ \pi_{s_1} | A : A \to X$. If we denote by $S_q(X)$ the VS of singular q-chains in X, then the homomorphisms $S_q(g) : S_q(A) \to S_q(X)$ and $S_q(i) : S_q(A) \to S_q(X)$ are chain homotopic. Hence we have a homomorphism $D : S_q(A) \to S_{q+1}(A)$ such that $\partial D + D\partial = S_q(g) - S_q(i)$. Thus $\partial D\tilde{c} = S_q(g)(\tilde{c}) - \tilde{c}$ and, hence, $[S_q(g)(\tilde{c})] = c$. But $[S_q(g)(\tilde{c})] = [S_q(i_{s_1})S_q(\pi_{s_1})(\tilde{c})] = H_q(i_{s_1})[S_q(\pi_{s_1})(\tilde{c})]$.

Now, let $[c] = H_q(i_s)(c_s) = H_q(i_t)(c_t) = [c']$. There is a q+1-chain d such that $c - c' = \partial d$. Similarly as above, we show the existence of a chain homomorphism $\varphi : S_q(A) \to S_q(X_u)$ where $A = \operatorname{supp} d, u \geq s,t$, such that $\partial\varphi d = \varphi\partial d = \varphi(c - c') = S_q(i_{su})(c_s) + - S_q(i_{tu})(c_t)$, which proves our assertion completely. q.e.d.

From now on, we shall consider only filtrations of a space X which are regular over any compact subset of X.

III. A-mappings

7. Let $(Y,\mathcal{Y})$ and $(X,\mathcal{X})$ be uniform spaces with filtrations $\{Y_s\}_{s\in S}$ and $\{X_s\}_{s\in S}$, respectively. We say that a mapping $f : Y \to X$ is an admissible mapping (A-mapping) w.r.t. $\{Y_s\},\{X_s\}$ if, for each $V \in \mathcal{X}$, there is $s_o \in S$ such that $f(Y_s) \subset V(X_s)$ for $s \geq s_o$. We shall say that f is a strong A-mapping if, for any $V \in \mathcal{X}$, there are $W \in \mathcal{Y}$ and $s_o \in S$ such that $f(W(Y_s)) \subset V(X_s)$ for $s \geq s_o$. Observe that if $f : Y \to X$ is a uniformly continuous A-mapping, then

f is a strong A-mapping. Moreover, if $(Z,\mathcal{Z})$ is a space with a filtration $\{Z_s\}_{s\in S}$ and $g : X \to Z$ is a strong A-mapping, then, for any A-mapping $f : Y \to X$, the superposition $g \circ f : Y \to Z$ is an A--mapping w.r.t. $\{Y_s\}$ and $\{Z_s\}$.

(7.1) Example. (i) Any compact mapping (i.e. such that $\mathrm{cl}\, F(X)$ is compact) is an A-mapping w.r.t. arbitrary filtrations $\{Y_s\}$, $\{X_s\}$ in Y and X, respectively.

(ii) Let E be an LCTVS filtrated by an increasing family of vector subspaces. Any linear combination, with coefficients being bounded scalar functions, of A-mappings $X \to E$ is, again, an A-mapping.

(iii) Let $L : \mathrm{dom}\, L \to F$, where $\mathrm{dom}\, L$ is a vector subspaces of E and F is an LCTVS filtrated, similarly as in (ii), by $\{F_s\}_{s\in S}$, be a linear and continuous Fredholm operator of index $k \geq 0$ such that $\mathrm{Im}\, L = F$. There is an increasing family $\{E_s\}_{s\in S}$ of linear subspaces of E, creating a filtration in E, such that L and any (nonlinear) L-compact mapping $f : E \to F$ are A-mappings w.r.t. $\{E_s\}$, $\{F_s\}$. For the proof, see [5].

(iv) Several, more concrete examples of A-mappings arise quite naturally when studying integral or ordinary differential equations (see [6], [7]).

(7.2) Let $(X,\mathcal{X})$ be a space with filtration $\{X_s\}_{s\in S}$ which is regular over a subset $Z \subset X$. If $f : X \to X$ is an A-mapping such that $f(X) \subset Z$, then, for any $q \geq 0$, $H_q(f)$ is decomposable w.r.t. the direct system $\varepsilon_q = \{H_q(X_s), H_q(i_{st})\}_{s\in S}$.

Proof. Let $U \in \mathcal{X}$. By the definition, there are $V \in \mathcal{X}$ and $s_o \in S$ such that $V \subset U$ and, for $t \geq s_o$, there is a mapping $\pi_t : V(Z) \cap V(X_t) \to X_t$ such that $\pi_t(x) = x$ for $x \in V(Z) \cap X_t$. There are symmetric vicinities $W, V' \in \mathcal{X}$, $W \subset V'$, $V' \circ V' \subset V$, $s_1 \geq s_o$ and sequences $\{\pi'_s : W(X_s) \cap W(Z) \to X_s\}_{s\geq s_1}$, $\{h'_s : [W(X_s) \cap W(Z)] \times [0,1] \to X\}_{s\geq s_1}$ such that $\pi'_s(x) = x$ for $x \in X_s \cap W(Z)$, $h'_s(x,0) = i_s \circ \pi'_s(x)$ and $h'_s(x,1) = x$, for $s \geq s_1$. Moreover, we know that, for $s \geq s_1$, h'_s is a V'-homotopy. Let $s_2 \geq s_1$ be such that, for $s \geq s_2$, $f(X_s) \subset W(X_s)$. Define $f_s = \pi'_s \circ f : X_s \to X_s$ for $s \geq s_2$. Observe that, for $t \geq s \geq s_2$, $i_{st} \circ f_s : X_s \to X_t$ and

$f_t \circ i_{st} : X_s \to X_t$ are homotopic to each other. Indeed, a homotopy $h : X_s \times [0,1] \to X_t$ given by the formula

$$h(x,t) = \begin{cases} \pi_t \circ h'_s(f(x),2t) & x \in X_s,\ t \in [0,1/2], \\ \pi_t \circ h_t(f(x),2-2t) & x \in X_s,\ t \in [1/2,1] \end{cases}$$

joins $i_{st} \circ f_s$ and $f_t \circ i_{st}$. Thus $\{H_q(f_s)\}_{s \geq s_2}$ is an endomorphism of the direct system $\varepsilon'_q = \{H_q(X_s), H_q(i_{st})\}_{s \geq s_2}$. By (6.3), we gather that the pair $(H_q(X), \{H_q(i_s)\}_{s \geq s_2})$ is compatible with ε'_q and $(H_q(i_s))_{s \geq s_2} : \lim\limits_{s \geq s_2} \varepsilon'_q \to H_q(X)$ is an isomorphism. At last, since $S' = \{s \in S | s \geq s_2\}$ is cofinal with S, $f \circ i_s = f|X_s$ and $i_s \circ f_s$ are (even V'-) homotopic to each other and, hence, the following diagram

$$\begin{array}{ccc} H_q(X_s) & \xrightarrow{H_q(i_s)} & H_q(X) \\ {\scriptstyle H_q(f_s)} \downarrow & & \downarrow {\scriptstyle H_g(f)} \\ H_q(X_s) & \xrightarrow[H_q(i_s)]{} & H_q(X) \end{array}$$

commutes for $s \geq s_2$, we gather that $\mathcal{D}_{U,q} = \{\varepsilon'_q, \{H_q(i_s)\}_{s \in S'}, \{H_q(f_s)\}_{s \in S'}\}$ is the wanted decomposition of $H_q(f)$ w.r.t. ε_q. Observe that the choice of s_2 in the above proof does not depend on $q \geq 0$. q.e.d.

(7.3) <u>Suppose that</u> X, $\{X_s\}_{s \in S}$, $Z \subset X$, $f : X \to X$ <u>satisfy the assumptions of</u> (7.2). <u>For any</u> $U \in \mathcal{X}$, <u>there is</u> $U' \in \mathcal{X}$ <u>such that, for each</u> T, $W \subset U'$, $\mathcal{D}_{T,q} = \mathcal{D}_{W,q}$.

P r o o f. Let V, $s_o \in S$, π_t for $t \geq s_o$ be as in the proof of (7.2). We take $U' \in \mathcal{X}$ such that $U' \circ U' \subset V$. Let $T, W \subset U'$ and let, for $s_2, \bar{s}_2 \geq s_o$, $\mathcal{D}_{W,q} = \{\varepsilon'_q, \{H_q(i_s)\}_{s \geq s_2}, \{H_q(f_s)\}_{s \geq s_2}\}$, $\mathcal{D}_{T,q} = (\varepsilon''_q, \{H_q(i_s)\}_{s \geq \bar{s}_2}, \{H_q(\bar{f}_s)\}_{s \geq s_2})$. We know that $f \circ i_s$ and $i_s \circ f_s$, for $s \geq s_2$, are W-homotopic and $f \circ i_s$, $i_s \circ \bar{f}_s$, for $s \geq \bar{s}_2$, are T-homotopic to each other, too. Let $s_3 \geq s_2, \bar{s}_2$. For $t \geq s_3$, let $h_t : X_t \times [0,1] \to X$ be a W-homotopy such that $h_t(\cdot,0) = i_t \circ f_t$ and $h_t(\cdot,1) = f \circ i_t$, let $g_t : X_t \times [0,1] \to X$ be a T-homotopy such that $g_t(\cdot,0) = i_t \circ \bar{f}_t$ and $g_t(\cdot,1) = f \circ i_t$. Let

$k_t : X_t \times [0,1] \to X$ be given by the formula

$$k_t(x,a) = \begin{cases} \pi_t \circ h_t(x,2a) & x \in X_t, \quad a \in [0,1/2], \\ \pi_t \circ g_t(x,2-2a) & x \in X_t, \quad a \in [1/2,1]. \end{cases}$$

We see that k_t is a homotopy joining f_t and $\bar{f}_t$. q.e.d.

IV. Lefschetz mappings

8. Let $(X,\mathcal{X})$ be a space. We say that a mapping $f : X \to X$ is a generalized Lefschetz mapping (generalized L-mapping) if there exists a filtration $\{X_s\}_{s\in S}$ of X (regular over compact subsets of X) which is regular over $f(X)$, f is an A-mapping w.r.t. $\{X_s\}$ and, for each $U \in \mathcal{X}$, there is $V \in \mathcal{X}$, $V \subset U$, such that $\mathcal{D}_V = \{\mathcal{D}_{V,q}\}_{q\geq 0}$ is an L-decomposition of $H(f)$ w.r.t. $\varepsilon = \{\varepsilon_q\}_{q\geq 0}$. For such a mapping, we define the generalized Lefschetz number

$$\Lambda(f,\{X_s\}_{s\in S}) = \lim_V \Lambda(H(f),\mathcal{D}_V)$$

where the limit is taken w.r.t. the net of elements of $\mathcal{X}$, directed by the inverse inclusion. The generalized Lefschetz number of f w. r.t. $\{X_s\}_{s\in S}$ is well-defined in view of (7.3).

(8.1) Example. If X is a Borsuk space (e.g. an ANR (metric)) and $f : X \to X$ is a compact mapping, then f is a generalized Lefschetz mapping. This is a simple consequence of the results from [4].

The next important example is given in teh following proposition.

(8.2) *Let $(X,\mathcal{X})$ be a space with a filtration $\{X_s\}_{s\in S}$ regular over compact subsets of X. Let $Z \subset X$ be such that:*

(i) $\{X_s\}_{s\in S}$ is regular and satisfies the condition (R) over Z,

(ii) there exists $W \in \mathcal{X}$ such that, for $s \geq s_o \in S$, $W(Z)\cap X_s$ is contained in a compact subset Z_s of X_s.

Any A-mapping $f : X \to X$ such that $f(X) \subset Z$ is a generalized L-mapping. Let $K = Q$. If $\Lambda(f,\{X_s\}) \neq 0$, then f has an approximate fixed point, i.e. for any $V \in \mathcal{X}$, there is $x \in X$ such that $(f(x),x) \in V$.

Proof. Take a vicinity $U \in \mathcal{X}$ such that $U \circ U \subset T \cap W$. There exists $s_2 \in S$ such that, for $q \geq 0$, $\mathcal{D}_{U,q}$ constructed in (7.2) (see the proof of (7.2)) is a decomposition of $H_q(f)$ w.r.t. ε_q.

Let $s_3 \geq s_o, s_1, s_2$ where s_1 comes from the formulation of the condition (R), and let $s \geq s_3$. We shall show that $\{H_q(f_s)\}_{q\geq 0}$ is an L-endomorphism of $\{H_q(X_s)\}_{q\geq 0}$. First, we observe that $f_s(X_s) \subset \subset U(Z)$ and $\mathrm{cl}\ U(Z) \subset U \circ U(Z) \subset T(Z)$ and $\mathrm{cl}\ U(Z) \subset W(Z)$. Hence, $A_s = \mathrm{cl}\ U(Z) \cap X_s \subset Z_s$ is compact and $A_s \subset T(Z) \cap X_s = Y_s$. By (R), there exist a convex subset C_s of an LCTVS E_s, an open subset G_s of C_s and mappings $r_s : G_s \to Y_s$, $j_s : Y_s \to G_s$ such that $r_s \circ j_s = \mathrm{id}_{Y_s}$. The following diagram

(**)
$$\begin{tikzcd} G_s \arrow[d, "g_s"'] \arrow[dr, "f_s r_s"] & Y_s \arrow[l, "j_s"'] \arrow[r, hook] \arrow[d, "f_s|Y_s"] & X_s \arrow[d, "f_s"] \arrow[dl, "f_s"'] \\ G_s & Y_s \arrow[l, "j_s"] \arrow[r, hook] & X_s \end{tikzcd}$$

where $g_s = j_s \circ f_s \circ r_s$, is commutative and $g_s(G_s) \subset j_s(A_s)$. Using the technique of Schauder's projection (see [3]) we establish the existence of a finite, compact polyhedron P_s such that $j_s(A_s) \subset P_s \subset G_s$ and a mapping $\bar{g}_s : G_s \to P_s$ which is homotopic to g_s, hence $H_q(g_s) = H_q(\bar{g}_s)$ for any $q \geq 0$. The following diagram is commutative

$$\begin{tikzcd} H_q(P_s) \arrow[r] \arrow[d, "H_q(\bar{g}_s|P_s)"'] & H_q(G_s) \arrow[d, "H_q(\bar{g}_s)"] \arrow[dl, "H_q(\bar{g}_s)"'] \\ H_q(P_s) \arrow[r] & H_q(G_s). \end{tikzcd}$$

Since $\dim_K H_q(P_s) < \infty$ for all $q \geq 0$ and $H_q(P_s) = 0$ for almost all q, we gather that $H_q(\bar{g}_s|P_s)$ is an L-endomorphism, hence, by (2.1) (i), $H_q(\bar{g}_s)$ and $H_q(g_s)$ are L-endomorphisms, too, and $\{H_q(G_s)\}_{q\geq 0}$ is of finite type. Passing to the homological analogue of (**), by (2.1) (i), we gather that $\{H_q(f_s)\}_{q\geq 0}$ is an L-endomorphism of a graded VS $\{H_q(X_s)\}_{q\geq 0}$.

The last part of the theorem follows easily, if $K = Q$ (the field of rational numbers), from Granas' version of the famous Lefschetz-Hopf theorem. Indeed, let $V \in \mathcal{X}$. Take a symmetric $W \in \mathcal{X}$ such that $W \circ W \subset V$. If $\Lambda(f,\{X_s\}) \neq 0$ and W is sufficiently small we know that $\Lambda(H(f),\mathcal{D}_W) \neq 0$. So, for sufficiently large s, $\Lambda(H(f_s)) \neq 0$. By [4], this means that there is a point $x \in X_s$ such that $f_s(x) = x$. Thus $(f(x), f_s(x)) = (f(x), x) \in W \circ W \subset V$. q.e.d.

The next result is related to the fixed point theory of compact mappings.

(8.3) <u>Observe that if</u> X, $\{X_s\}_{s\in S}$ <u>satisfy the assumptions of</u> (8.2) <u>and</u> $f: X \to X$ <u>is a compact mapping, then</u> f <u>is a generalized</u> L-<u>mapping, and if</u> $\Lambda(f,\{X_s\}) \neq 0$, <u>then</u> f <u>has a fixed point. More generally, if all the assumptions of</u> (8.2) <u>are satisfied and</u> f <u>satisfies the so-called Palais-Smale condition, i.e.</u>

(PS) $\forall V \in \mathcal{X} \quad \exists x \in X \quad (f(x),x) \in V \implies \exists x_o \in X\ f(x_o)=x_o$,

<u>then we shall obtain the existence of fixed points.</u>

Thus, we see that our algebraic setting is applicable for spaces which, in some sense, are more general than Borsuk ones (e.g. ANRs (metric)).

As a simply corollary we get:

(8.4) <u>Suppose</u> X, $\{X_s\}_{s\in S}$, $Z \subset X$ <u>satisfy the assumptions of</u> (8.2). <u>Let</u> $K = Q$. <u>If, for any</u> $s \geq s_1$, X_s <u>is acyclic, then any</u> A-<u>mapping</u> $f: X \to X$ <u>such that</u> $f(X) \subset Z$ <u>has an approximate fixed point.</u>

Recall that a space is called <u>acyclic</u> if $H_o(X) = Q$ and $H_q(X) = 0$ for $q > 0$.

The next results seem to be most interesting.

(8.5) <u>Let</u> X, $\{X_s\}_{s\in S}$, $Z \subset X$ <u>be as in</u> (8.2) <u>and let</u> $f: X \to X$ <u>be a uniformly continuous</u> A-<u>mapping such that</u> $\{X_s\}$ <u>is regular over</u> $f(X)$ <u>and, for some positive integer</u> n, $f^n(X) \subset Z$. <u>Then there exist an open subset</u> G <u>of</u> X <u>and</u> $V \in \mathcal{X}$ <u>such that</u> $f^n(X) \subset \subset \bar{G}$ <u>and</u> $V(f(G)) \subset G$. <u>Moreover, we claim that</u> f <u>is a generalized Lefschetz mapping, and if</u> $\Lambda(f,\{X_s\}) \neq 0$, <u>then</u> f <u>possesses an approximate fixed point, provided</u> $K = Q$. <u>Additionally, if, for</u> $s \geq s_1$, X_s <u>is acyclic, then any</u> A-<u>mapping with the above-mentioned properties has such a fixed point.</u>

P r o o f. Let $U \in \mathcal{X}$, $U \subset T \cap W$. Take a vicinity (open) W_n such that $W_n \circ W_n \subset U$. Next, we take vicinities $V', W_1, W_2, \ldots, W_{n-1}$ such that $V' \subset W_1 \subset W_2 \subset \ldots \subset W_{n-1} \subset W_n$ and $V' \circ W_i \subset W_{i+1}$ for $i = 1,2,\ldots,n-1$. We define $G = W_n(Z) \cap f^{-1}(W_{n-1}(Z)) \cap \ldots \cap f^{-n+1}(W_1(Z))$. Since f^i for any i is uniformly continuous, there is $V_i \in \mathcal{X}$ such that, for any $y, z \in X$, if $(y,z) \in V_i$, then $(f^i(y), f^i(z)) \in V'$.

Let $V = \bigcap_{i=0}^{n} V_i$. Now, let $z \in V(f(G))$. Then, there is $y \in f(G)$ such that $(y,z) \in V$. Since $y \in f(G) \subset W_{n-1}(Z) \cap \ldots \cap f^{-n+2}(W_1(Z)) \cap \cap f^{-n+1}(Z)$, therefore, for any $i = 0,1,\ldots,n-1$, $y \in f^{-i}(W_{n-1-i}(Z))$ where $W_o = \Delta_X$, $f^o = \mathrm{id}$. So, for $i = 0,1,\ldots,n-1$, there is $a_i \in Z$ such that $(a_i, f^i(y)) \in W_{n-1-i}$. Since $(f^i(y), f^i(z)) \in V'$, we gather that $(a_i, f^i(z)) \in V' \circ W_{n-1-i} \subset W_{n-i}$. Thus $z \in f^{-i}(W_{n-i}(Z))$ for any $i = 0,1,\ldots,n-1$. This shows that $z \in G$.

Observe that G is filtrated by $\{G_s = G \cap X_s\}_{s\in S}$ in view of (6.1), and that the filtration $\{G_s\}$ is regular over any compact subset of G, it is regular and satisfies (R) over $f(G)$, since any open subset of a Borsuk space is again a Borsuk space.

Now, we construct a sequence $U_o, U_o', U_1, U_1', \ldots, U_n, U_n' \subset V$ of elements of $\mathcal{X}$ such that $U_o \circ U_i' \subset U_{i+1}$, $U_i \subset U_i'$ for $i = 0,1,\ldots,$ $n-1$, and such that, for $(x,y) \in U_i$, $(f(x),f(y)) \in U_i'$. Let $s_o \in S$ and $\mathcal{D}_{U_o} = \{\mathcal{D}_{U_o,q}\}_{q\geq 0}$, where $\mathcal{D}_{U_o,q} = (\varepsilon_q, \{H_q(i_s)\}_{s\geq s_o}, \{H_q(f_s)\}_{s\geq s_o})$ be a decomposition of $H(f)$ w.r.t. $\varepsilon = \{\varepsilon_q\}_{q\geq 0}$ (see (7.2)). We know that, for any $s \geq s_o$, $(f(x), f_s(x)) \in U_o \subset U_1$. By induction we prove that $(f^i(x), f_s^i(x)) \in U_i \subset V$ for $i = 0,1,\ldots,n$. Hence, for $s \geq s_o$, $i = 0,1,\ldots,n-1$, $f_s^{-i+1}(G_s) \subset f_s^{-i}(G_s)$ and $f_s : f_s^{-i}(G_s) \to f_s^{-i}(G_s)$ for $i = 0,1,\ldots,n$. Taking a sufficiently small U_o we may assume that, in view of (8.2), $\mathcal{D}_{U_o} \mid G = \{\mathcal{D}_{U_o,q} \mid G\}_{q\geq 0}$ where $\mathcal{D}_{U_o,q} \mid G = (\{H_q(G_s), H_q(i_{st} \mid G_s)\}_{s\geq s_o}, \{H_q(i_s \mid G_s)\}_{s\geq s_o}, \{H_q(f_s \mid G_s)\}_{s\geq s_o})$ is an L-decomposition of $\{H_q(f \mid G)\}_{q\geq 0}$. Now, look at the following diagram $(s \geq s_o)$.

$$\begin{tikzcd}
G_s \arrow[r, hook] \arrow[d, "f_s"] & f_s^{-1}(G_s) \arrow[r, hook] \arrow[d, "f_s"] \arrow[dl, "f_s"'] & f_s^{-2}(G_s) \arrow[r, hook] \arrow[d, "f_s"] \arrow[dl, "f_s"'] & \cdots \arrow[r] \arrow[dl] & f_s^{-n+1}(G_s) \arrow[r, hook] \arrow[d, "f_s"] \arrow[dl] & f^{-n}(G_s) = X_s \arrow[d, "f_s"] \arrow[dl, "f_s"'] \\
G_s \arrow[r, hook] & f_s^{-1}(G_s) \arrow[r, hook] & f_s^{-2}(G_s) \arrow[r, hook] & \cdots \arrow[r] & f_s^{-n+1}(G_s) \arrow[r, hook] & f_s^{-n}(G_s) = X.
\end{tikzcd}$$

It is commutative, so, by applying (2.1) (i) several times to the adequate homology diagram we get that $\mathcal{D}_{U_o}$ is an L-decomposition of $\{H_q(f)\}_{q\geq 0}$. The last part is rather obvious. q.e.d.

The following theorem has connections with the asymptotic fixed point theory of compact mappings.

(8.6) Let B denote the unit ball in a normed space E. Let $\{B_s\}_{s\in S}$ be a filtration of B of the form $B_s = B \cap E_s$, where E_s, for $s \in S$, is a finite-dimensional vector subspace of E and $\mathrm{cl}\,(\bigcup_{s\in S} E_s) = E$. Any uniformly continuous A-mapping such that, for some positive integer n, $f^n(X)$ is compact, has a fixed point.

P r o o f. By (6.2) (ii), the filtration $\{B_s\}_{s\in S}$ is regular and satisfies (R). Since, for any $s \in S$, B_s is acyclic, we get the assertion. q.e.d.

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