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A note on Fiedler-Moravek combinatorial problem

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A NOTE ON FIEDLER - MORÁVEK COMBINATORIAL PROBLEM*)

Jiří Vinárek

M.Fiedler and J.Morávek have formulated in [1] the following:

1.Problem. Let $A_1,\dots,A_n$ be vertices of a convex n-gon, $\underline{E}_2$ be the Euclidean plane. Find the smallest number K(n) of convex sets $\underline{S}_1,\dots,\underline{S}_{K(n)}$ such that

$$\underline{M} = \underline{E}_2 \setminus \{A_1,\dots,A_n\} = \bigcup_{i=1}^{K(n)} \underline{S}_i .$$

We are going to prove the following :

Hypothesis. (J.Kratochvíl) If we consider only pairwise disjoint partitions of $\underline{M}$ then the smallest number $k(n) = \lceil \frac{2}{3} n \rceil + 1$.

2.Lemma. Boundaries of parts $\underline{S}_1,\dots,\underline{S}_{k(n)}$ are unions of straight lines, half-lines and abscissas.

Proof. If $X,Y \in \mathrm{bd}\, \underline{S}_i \cap \mathrm{bd}\, \underline{S}_j$ then $X,Y \in \mathrm{cl}\, \underline{S}_i \cap \mathrm{cl}\, \underline{S}_j$. Since $\underline{S}_i$, $\underline{S}_j$ are convex, their closures $\mathrm{cl}\, \underline{S}_i$, $\mathrm{cl}\, \underline{S}_j$ are convex as well. Hence, the abscissa $XY \subset \mathrm{cl}\, \underline{S}_i \cap \mathrm{cl}\, \underline{S}_j$ and also $XY \subset \mathrm{bd}\, \underline{S}_i \cap \mathrm{bd}\, \underline{S}_j$, q.e.d.

3.Definitions. a) Let $\mathcal{S} = \{\underline{S}_1,\dots,\underline{S}_k\}$ be a partition of $\underline{M}$ (i.e. $M = \bigcup_{i=1}^{k} \underline{S}_i$, $\underline{S}_i \cap \underline{S}_j = \emptyset$ for $i \neq j$), $X \in \underline{E}_2$. Then a *degree* of X with respect to $\mathcal{S}$ is defined by $\deg(X,\mathcal{S}) = |\{i \mid X \in \mathrm{cl}\, \underline{S}_i\}|$.

b) A straight line (or its subset) p is called an *edge of the partition* $\mathcal{S}$ if there exist i,j such that $p \subset \mathrm{cl}\, \underline{S}_i \cap \mathrm{cl}\, \underline{S}_j$ and for any straight line, abscissa or half-line $q \supset p$ with $q \subset \mathrm{cl}\, \underline{S}_i \cap \mathrm{cl}\, \underline{S}_j$ there is $q = p$.

c) A point X is called a *vertex of the partition* $\mathcal{S}$ iff it is an end point of some edge of $\mathcal{S}$. It is called a *proper vertex* if $\deg(X,\mathcal{S}) \geq 3$.

4.Proposition. Let $\mathcal{S} = \{\underline{S}_1,\dots,\underline{S}_k\}$ be a partition of $\underline{M}$, V be a vertex

*) This paper is in final form and no version of it will be submitted for publication elsewhere.

of $\mathcal{S}$, $\deg(V,\mathcal{S}) = d \geq 4$. Then there exists a partition $\mathcal{D} = \{\underline{D}_1,\dots,\underline{D}_{k'}\}$ of $\underline{M}$ such that $k' \leq k$, $\deg(V,\mathcal{D}) = d-1$ and there is a bijection $f : \underline{E}_2 \longrightarrow \underline{E}_2$ such that $\deg(f(X),\mathcal{D}) \leq \deg(X,\mathcal{S})$ or $\deg(f(X),\mathcal{D}) \leq 3$, for any $X \in \underline{E}_2$.

<u>Proof.</u> Let $p_1,\dots,p_d$ be edges of $\mathcal{S}$ containing V. One can suppose that the angle $\measuredangle p_i p_{i+1}$ between p_i and p_{i+1} contains no other p_j. The Dirichlet principle implies that there exists i such that $\measuredangle p_i p_{i+2} \leq$ $\leq 180^\circ$. Suppose that $p_{i+1} \subset \mathrm{bd}\, \underline{S}_q \cap \mathrm{bd}\, \underline{S}_r$, $q < r$.

Consider the following cases :

(i) p_{i+1} is a half-line

(ii) $p_{i+1} = VW$ with $\deg(W,\mathcal{S}) \geq 3$

(iii) $p_{i+1} = VW$ with $\deg(W,\mathcal{S}) = 2$

In the case (i) there is $\underline{S}_q \cup \underline{S}_r$ also convex (see Fig.1) and one can define $\mathcal{D} = \{\underline{D}_1,\dots,\underline{D}_{k-1}\}$ where

$$\underline{D}_j = \underline{S}_j \text{ for } j < r,\ j \neq q$$
$$\underline{D}_j = \underline{S}_q \cup \underline{S}_r \text{ for } j = q$$
$$\underline{D}_j = \underline{S}_{j+1} \text{ for } j \geq r$$

If we put f as the identity mapping then $\mathcal{D}$, f satisfy assertions of Proposition.

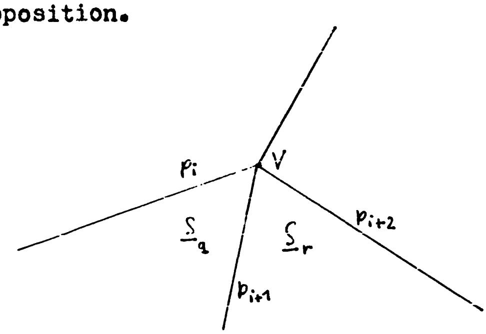

Fig. 1.

In the case (ii) there exists an edge p with an end-vertex W such that $\measuredangle pp_{i+1} < 180^\circ$. Without loss of generality one can suppose that $p \subset \mathrm{cl}\, \underline{S}_q$. Then one can choose $V' \in p_{i+2}$ such that the angle between p and WV' is less than 180° and V' is not a vertex of $\mathcal{S}$ (see Fig.2). Now one can define $\underline{D}_q$ as a union of $\underline{S}_q$ and the triangle $\underline{T}$ with vertices V, V', W, $\underline{D}_r = \underline{S}_r \setminus \underline{T}$, $\underline{D}_j = \underline{S}_j$ for any $j \neq q,r$. $\mathcal{D} = \{\underline{D}_1,\dots,\underline{D}_k\}$ is the asked partition of $\underline{M}$. (Actually, the only new vertex is V' with $\deg(V',\mathcal{D}) = 3$ and we can put f as the identity mapping.)

In the case (iii) one can suppose that $W \in \{A_1,\dots,A_n\}$. Consider three cases :

(a) There exists a straight line m containing W such that the half-plane mV contains the n-gon $A_1 \dots A_n$ (see Fig.3).

One can suppose that m contains no vertex X of $\mathcal{S}$ such that $X \neq W$. Denote by $\widetilde{mV}$ the union of the open half-plane mV and the right half-line $m^+ \subset m$ with the end-point W.

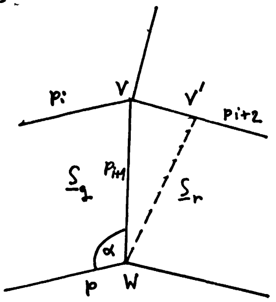

Fig.2

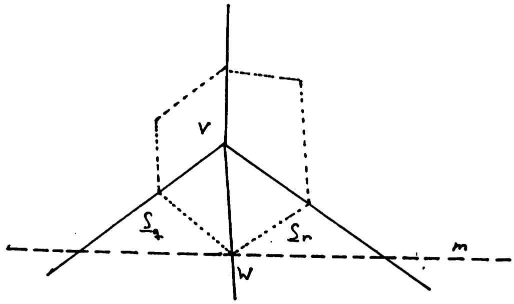

Fig.3

Then define for any $j \neq q,r$: $\underline{D}_j = \underline{S}_j \cap \widetilde{mV}$. Further define : $\underline{D}_r = \underline{E}_2 \setminus \widetilde{mV} \setminus \{W\}$, $\underline{D}_q = (\underline{S}_q \cup \underline{S}_r) \cap \widetilde{mV}$. Clearly, $\mathfrak{D} = \{\underline{D}_1, \dots, \underline{D}_k\}$ is a convex partition of $\underline{M}$, $\deg(V, \mathfrak{D}) = d-1$. One can put f as the identity mapping.

(b) Non(a) and cl $\underline{S}_q \cup$ cl $\underline{S}_r$ is convex. Then choose a line m such that the only vertex of $\mathcal{S}$ lying on m is W (see Fig.4). Denote by m^+ (m^-, resp.) the open half-line of m with end-point W which intersects $\underline{S}_r$ ($\underline{S}_q$, resp.). Then define $\widetilde{mV}$ as the union of the open half-plane mV and m^+. Further put :

$$\underline{D}_j = \underline{S}_j \text{ for } j \neq q,r$$
$$\underline{D}_q = (\underline{S}_q \cup \underline{S}_r) \cap \widetilde{mV}$$
$$\underline{D}_r = (\underline{S}_q \cup \underline{S}_r) \setminus \widetilde{mV} \cup (m^- \cap \mathrm{cl}(\underline{S}_q \cup \underline{S}_r))$$

Clearly, $\mathfrak{D} = \{\underline{D}_1, \dots, \underline{D}_k\}$ is a convex partition of $\underline{M}$ and $\deg(V, \mathfrak{D}) = d - 1$.

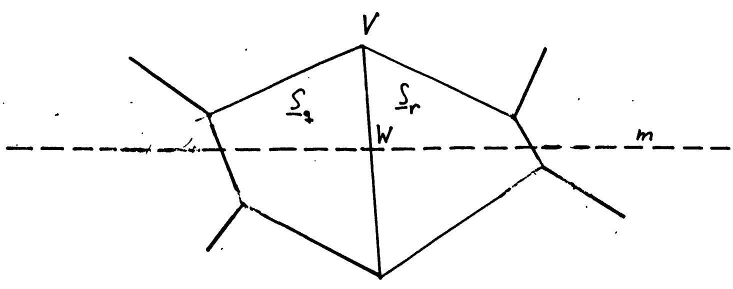

Fig.4.

One can again put f as the identity mapping.

(c) Non (a) and cl $\underline{S}_q \cup$ cl $\underline{S}_r$ is not convex (see Fig.5). Then the half-line VW contains another vertex U of $\mathcal{S}$. If $U \in \{A_1, \dots, A_n\}$

then there exists a tangent t to n-gon at U. If $U \in \mathrm{cl}\,\underline{S}_u$, $u \neq q,r$ then one can define $\underline{S}'_u$ as the open half-plane opposite to tW with the right half-line t^+ added, $\underline{S}'_j = \underline{S}_j \setminus \underline{S}'_u$ and then apply (b) since $\mathrm{cl}\,\underline{S}'_q \cup \mathrm{cl}\,\underline{S}'_r$ is convex.

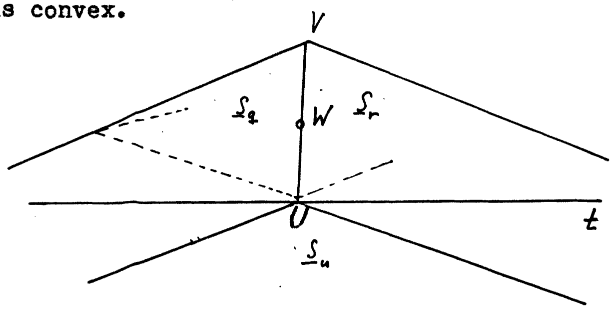

Fig. 5.

If $U \notin \{A_1,\dots,A_n\}$ is a point of the interior of the given n-gon, $U \in \mathrm{bd}\,\underline{S}_q \cap \mathrm{bd}\,\underline{S}_r \cap \mathrm{bd}\,\underline{S}_u$, $u \neq q,r$, $UU_1 \subset \mathrm{bd}\,\underline{S}_q \setminus \mathrm{bd}\,\underline{S}_r$, $UU_2 \subset \mathrm{bd}\,\underline{S}_r \setminus \mathrm{bd}\,\underline{S}_q$ are border lines such that $UU_1 \neq p_{i+1} \neq UU_2$. If there exists $A \in UU_2 \cap \{A_1,\dots,A_n\}$ then put $U_3 = A$ otherwise choose $U_3 \in UU_2$ arbitrarily. Then define a point $V' \in p_i$ as the intersection of p_i and U_3W and U' as the point of intersection of lines $V'U_3$ and U_1U (see Fig.6). Further put U_2' as the point of intersection of bd $\underline{S}_u$ and $V'U'$ distinct from U_3 (see Fig.6). Now use points U', U_2' as new vertices of a partition (instead of U, U_2), connect U' (U_2', resp.) with any vertex X of $\mathcal{S}$, $X \neq V$ ($X \neq U$, resp.) such that UX (U_2X, resp.) is an edge of $\mathcal{S}$. Of course, connect also $U'V'$.

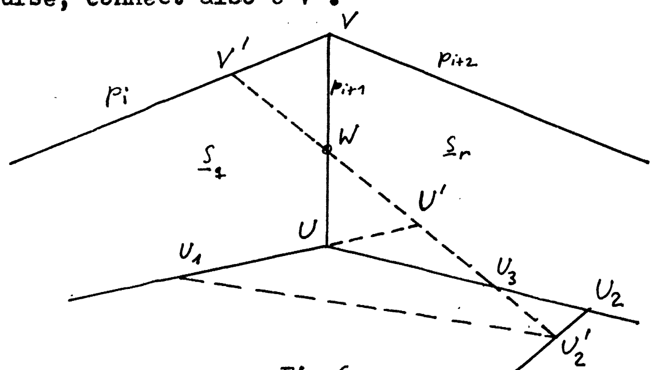

Fig.6.

The new partition $\mathcal{D}$ has again k elements, $\deg(U',\mathcal{D}) = \deg(U,\mathcal{S})$, $\deg(V,\mathcal{D}) = d - 1$, $\deg(U_3,\mathcal{D}) = 3$, $\deg(V',\mathcal{D}) = 3$ and $\deg(X,\mathcal{D}) = \deg(X,\mathcal{S})$ for any $X \neq V, V', U, U', U_2, U_2', U_3$. Put $f(U) = U'$, $f(U') = U$, $f(U_2) = U_2'$, $f(U_2') = U_2$, $f(X) = X$ for any $X \neq U, U', U_2, U_2'$.

One can check conditions of Proposition. Q.E.D.

5. Using this Proposition and the method of induction one can suppose that the given partition $\mathcal{S}$ of $\underline{M}$ has only vertices of degrees 2 and 3 (and that all vertices of degree 2 are vertices of the given n-gon). Let δ be the diameter of the set of vertices of $\mathcal{S}$ and let $\{p_1,\dots,p_s\}$ be the set of all half-line edges of $\mathcal{S}$. If $p_i = X_iY_i$ then denote by P_i the point of p_i such that $\rho(X_i,P_i) = \delta$. It is evident that all the vertices of $\mathcal{S}$ are situated inside the s-gon $\underline{G}$ with vertices $P_1,\dots,P_s$ (see Fig.7).

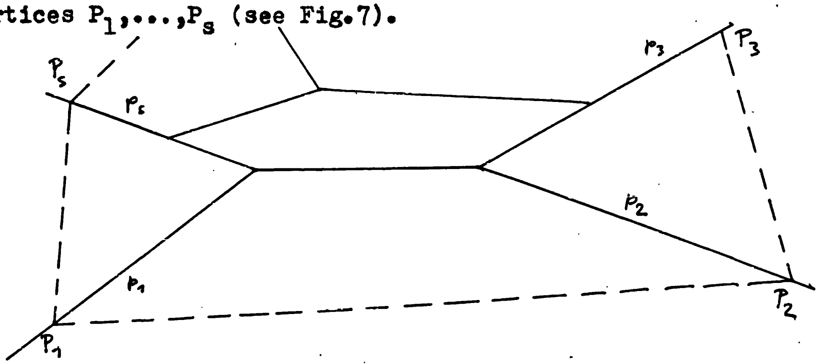

Fig.7.

Moreover, $\mathcal{S}$ induces a partition $\tilde{\mathcal{S}}$ of the interior of $\underline{G}$ with the same number of elements. So, it suffices to count the number k of elements of $\tilde{\mathcal{S}}$. Denote by $\tilde{v}$ the number of proper vertices of $\tilde{\mathcal{S}}$ (if v is the number of proper vertices of $\mathcal{S}$ then $\tilde{v} = v + s$ where s is the number of half-lines of $\mathcal{S}$), $\tilde{h}$ the number of edges of $\tilde{\mathcal{S}}$.

Euler formula implies that $k + \tilde{v} = \tilde{h} + 1$. Clearly, $\tilde{h} = \frac{3}{2}\tilde{v}$. Hence, $k = \frac{\tilde{v}}{2} + 1$. (*)

6. Our goal is to minimize $\tilde{v}$. We shall study the number adj X of proper vertices of $\tilde{\mathcal{S}}$ adjacent to a vertex X of the given n-gon. (If a vertex X is adjacent to two vertices A,B of $\tilde{\mathcal{S}}$ we shall count only $\frac{1}{2}$ of vertex X adjacent to A and $\frac{1}{2}$ of X adjacent to B etc). Of course, if $X \in \{A_1,\dots,A_n\}$ is a proper vertex of $\tilde{\mathcal{S}}$ then X is adjacent to X.

For vertices $X = A_i$, $Y = A_{i+1}$, $Z = A_{i+2}$ we have the following configurations :

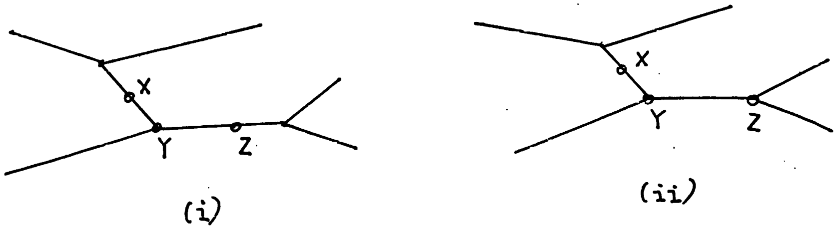

Fig.8a

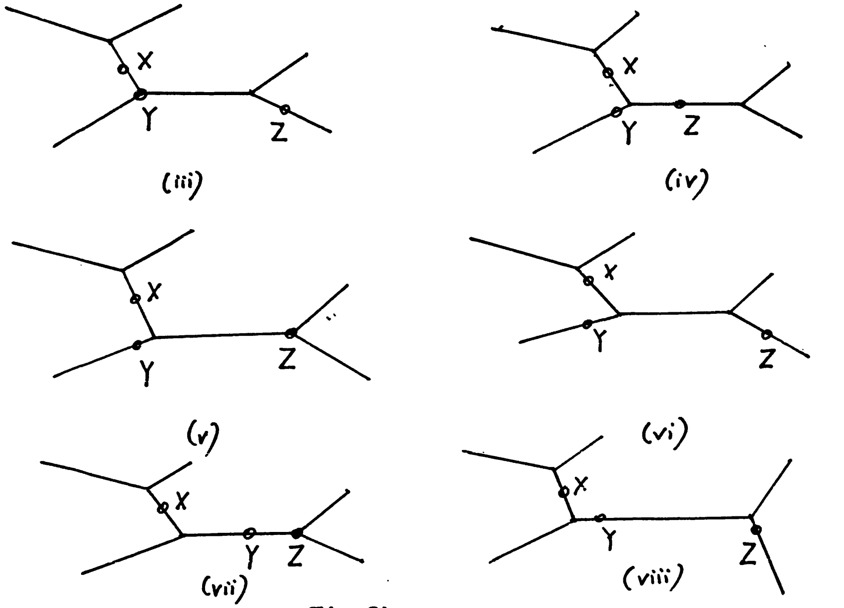

Fig. 8b.

In the first case (see Fig. 9) we have adj $X \geq 1$ (at least half-points A and B are adjacent to X), adj $Y = 2$ (adjacent points Y,C), adj $Z \geq 1$ (at least half-points D,E adjacent to Z).

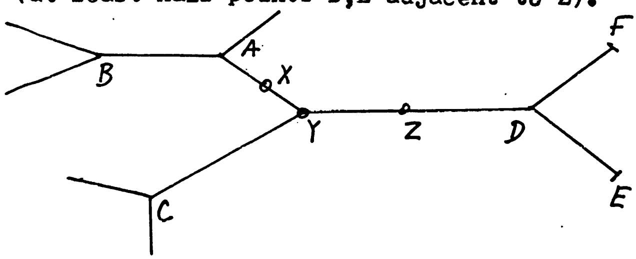

Fig. 9.

Similarly one can check the other configurations :

(ii) adj $X \geq 1$, adj $Y = 2$, adj $Z \geq 2$

(iii) adj $X \geq 1$, adj $Y = 2$, adj $Z \geq 2$

(iv) adj $X \geq \frac{4}{3}$, adj $Y = \frac{4}{3}$, adj $Z \geq \frac{4}{3}$

(v) adj $X \geq \frac{1}{2}$, adj $Y = \frac{3}{2}$, adj $Z \geq 2$

(vi) adj $X \geq \frac{3}{2}$, adj $Y = \frac{3}{2}$, adj $Z \geq \frac{3}{2}$

(vii) adj $X \geq 1$, adj $Y = 1$, adj $Z \geq 2$

(viii) adj $X \geq 1$, adj $Y = \frac{3}{2}$, adj $Z \geq \frac{3}{2}$

Hence, adj A_i + adj A_{i+1} + adj $A_{i+2} \geq 4$.

Since $\tilde{v} \geq \sum_{i=1}^{n}$ adj A_i there is $\tilde{v} \geq \lceil \frac{4}{3} n \rceil$. By ($*$) we have $k \geq \lceil \frac{2}{3} n \rceil + 1$,

Q.E.D.

7.Construction. One can construct a partition $\mathcal{S}$ of $\underline{M}$ as follows : for $j = 1,\dots,\lceil \frac{n}{3}\rceil$ denote by B_j the point of intersection of lines $A_{3j-2}A_{3j-1}$ and $A_{3j}A_{3j+1}$. Further define m_{2j-1} as an open half-line which is the axis of the exterior angle $\sphericalangle B_{j-1}A_{3j-2}B_j$, m_{2j} as a closed half-line which is the axis of the exterior angle $\sphericalangle A_{3j-2}B_jA_{3j+1}$, $\underline{C}_{2j-1}$ as the open set with the border lines $m_{2j-1}, A_{3j-2}B_j, m_{2j}$, $\underline{C}_{2j}$ as the open set with the border lines $m_{2j}, B_jA_{3j+1}, m_{2j+1}$. Finally define $\underline{D}_{2j-1} = \underline{C}_{2j-1} \cup m_{2j-1} \cup A_{3j-2}A_{3j-1}$ (as the open abscissa), $\underline{D}_{2j} = \underline{C}_{2j} \cup m_{2j} \cup A_{3j}A_{3j+1}$ (as the open abscissa), $\underline{D}_{\lceil\frac{2}{3}n\rceil+1} = \bigcup_{j=1}^{\lceil\frac{m}{3}\rceil} B_jA_{3j} \cup \bigcup_{j=1}^{\lceil\frac{m}{3}\rceil} A_{3j-1}B_j \cup \operatorname{int}\underline{P}$ where $\underline{P}$ is the polygon $A_1B_1A_4B_2\dots A_n$ (see Fig.10).

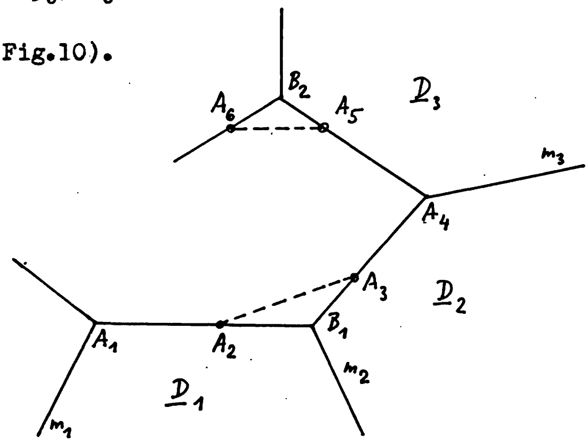

Fig.10.

One can check that $\mathcal{D} = \{\underline{D}_1,\dots,\underline{D}_k\}$ is the asked partition of $\underline{M}$.

8.Non-disjoint case. If one does not suppose the assumption of pairwise disjointness of a partition then generally $K(n) \neq k(n)$ – e.g. while $k(8) = 7$, $K(8) \leq 6$ (see Fig.11) :

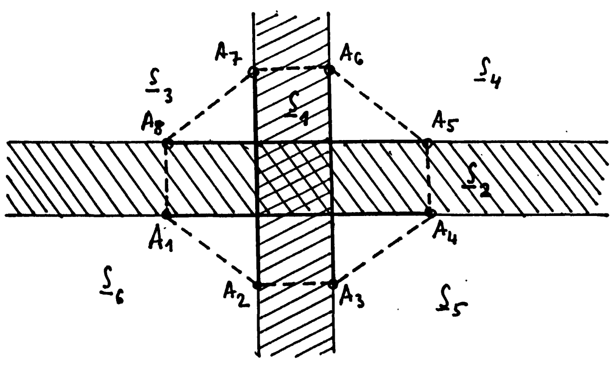

Fig. 11

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