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Properties of Measure and Category in Generalized Cohen's and Silver's Forcing

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We consider generalized Cohen's and Silver's forcing and characterize the following properties of such generic extensions: Every null (resp. meager) set of $M[G]$ is contained in a null (resp. meager) set coded in M; the set of reals from ground model is nonmeasurable in $M[G]$; ${}^{\omega}\omega \cap M$ is a dominating family in ${}^{\omega}\omega \cap M[G]$.

§ 0. **Introduction.** A natural generalization of Cohen's set of forcing conditions (two valued functions with domain a finite subset of ω) is the set of two valued functions with domain an element of an ideal J on ω. It is denoted by $C(J)$ and was investigated by S. Grigorieff in [5]. In § 3 we study the properties of generic extension obtained by this forcing in a dependence of the combinatorial properties of an ideal J investigated in § 1. Let M be a transitive model of ZFC, let $C(J) \in M$, and let $G \subseteq C(J)$ be a generic filter over M. We prove the next two theorems:

Theorem A. The following are equivalent

(i) J is an r^*-ideal

(ii) every null set of the Cantor space in $M[G]$ is covered by a null set coded in M.

(iii) ${}^{\omega}2 \cap M$ is not a null set in ${}^{\omega}2 \cap M[G]$.

Theorem B. The following are equivalent

(i) J is a regular p^*-ideal

(ii) $\forall f \in {}^{\omega}\omega \cap M[G] \quad \exists h \in {}^{\omega}\omega \cap M \quad \forall n \quad f(n) \leqq h(n)$

(iii) every meager set of the Cantor space in $M[G]$ is covered by a meager set coded in M.

Results of § 2 are needed only in the proofs of the propositions 3.3, 3.13. 3.16 and so can be omitted at the first reading. In § 4 we apply Grigorieff's characterization of generalized Silver's forcing to results of § 3.

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§ 1. Some properties of ideals and the question of their existence.

Throughout this paper an ideal on ω will mean an ideal containing the ideal of finite subsets of ω. We usually denote it by J. Then J^* will mean its dual filter and $J^+ = \mathscr{P}(\omega) - J$.

1.1. Definition. (i) A sequence $\{x_n; n \in \omega\}$ is a J-partition (resp. strong J-partition) if it is a partition of ω and no finite union of elements of the partition is in J^* (resp. in J^+).

(ii) An ideal J on ω is a p-ideal (resp. p^*-ideal) if for every J-partition (resp. strong J-partition) there exists $x \in J^+$ (resp. $x \in J^*$) and x meets each element of the partition at a finite set.

(iii) J is an r-ideal (resp. r^*-ideal) if for every J-partition (resp. strong J-partition) $\{x_n; n \in \omega\}$ there exists $x \in J^+$ (resp. $x \in J^*$) such that $|x \cap x_n| \leqq n$ for $n \in \omega$.

(iv) J is an s-ideal (resp. s^*-ideal) if for every J-partition (resp. strong J-partition) there exists $x \in J^+$ (resp. $x \in J^*$) such that $|x \cap x_n| \leqq 1$ for $n \in \omega$.

(v) J is regular if for every partition $\{x_n; n \in \omega\}$ of ω into finite sets there is an infinite set $a \subseteq \omega$ such that $\bigcup\{x_n; n \in a\} \in J$.

It is not hard to prove the following:

1.2. Proposition. J is a p-ideal (resp. p^*-ideal) if and only if for every decreasing sequence $\{x_n; n \in \omega\}$ of subsets of ω which are in J^+ (resp. in J^*) there is a set $x \in J^+$ (resp. $x \in J^*$) such that $x - x_n$ is finite for every n.

1.3. Lemma. (a) Every s-ideal is r-ideal,

(b) every r-ideal is p-ideal,

(c) every s^*-ideal is r^*-ideal,

(d) every r^*-ideal is p^*-ideal,

(e) every p-ideal which is also r^*-ideal is r-ideal,

(f) every p-ideal which is also s^*-ideal is s-ideal,

(g) every r^*-ideal is regular.

Proof. (a)–(f) follows from definition.

(g) Let $\{x_n; n \in \omega\}$ be a partition of ω into finite sets. Denote $g(n) = \sum_{i<n} (i + 1)$. Thus $g(n + 1) - g(n) > n$. The sets $u_n = \bigcup\{x_k; g(n) \leqq k < g(n + 1)\}$, $n \in \omega$, form a strong J-partition. Let $x \in J^*$ be such that $|x \cap u_n| \leqq n$. Then for every n there is $k \in \langle g(n), g(n + 1))$ such that $x \cap x_k = \emptyset$ and so x avoids infinitely many sets x_n.

1.4. Proposition. The following are equivalent.

(i) J is an r-ideal (resp. r^*-ideal);

(ii) there is a function $f \in {}^{\omega}\omega$ such that for every J-partition (resp. strong J-partition) $\{x_n; n \in \omega\}$ there is $x \in J^+$ (resp. $x \in J^*$) such that $|x \cap x_n| \leqq f(n)$ for $n \in \omega$;

(iii) for arbitrary nondecreasing unbounded function $h \in {}^{\omega}\omega$ and for every J-partition (resp. strong J-partition) $\{x_n; n \in \omega\}$ there is $x \in J^+$ (resp. $x \in J^*$) such that $|x \cap x_n| \leqq h(n)$ for $n \in \omega$.

Proof. It is enough to prove (ii)→(iii).

Let $\{x_n; n \in \omega\}$ be a J-partition (resp. strong J-partition). Define: $g(0) = 0$, $g(n) = \min\{k > g(n-1);\ h(k) \geqq f(n)\}$ for $n > 0$. The sets $y_n - \bigcup\{x_k: g(n) \leqq k < g(n+1)\}$ form a J-partition (resp. strong J-partition). Let $x \subseteq \omega$ be arbitrary such that $|x \cap y_n| \leqq f(n)$ for $n \in \omega$. For any $k \in \omega$ there is n such that $g(n) \leqq k < g(n+1)$ and $|x \cap x_k| \leqq |x \cap y_n| \leqq f(n) \leqq h(g(n)) \leqq h(k)$.

1.5. Lemma. Let J' be generated over J by a set $y \subseteq \omega$ (in this case we say that J' is one-generated over J). Then if J is a p-ideal (resp. r-ideal, resp. s-ideal) then J' is such too.

Proof. Let $\{x_n; n \in \omega\}$ be arbitrary J'-partition. The sets $y_0 = x_0 \cup y$, $y_n = x_n - y$ $n \geqq 1$ form a J-partition. If $z \in J^+$ is such that $z \cap y_n$ is finite for $n \in \omega$ then $y \cap z$ is finite and so $z \in (J')^+$. Put $x = z - y$. Then $x \in (J')^+$ and $|x \cap x_n| \leqq |z \cap y_n|$ for $n \in \omega$.

1.6. Lemma. Let J' be countably generated over J. Then if J is a p-ideal (resp. r-ideal, resp. s-ideal) then J' is such too.

Proof. Let $\{y_n, n \in \omega\}$ generates J' over J. Let $\{x_n; n \in \omega\}$ be arbitrary J'-partition. Since J is a p-ideal (see Proposition 1.2) there is a set $y \in \mathscr{P}(\omega) - J^*$ such that $x_n - y$, $y_n - y$, $n \in \omega$, are finite sets. The ideal J'' generated over J by y contains the ideal J' and the given partition is a J''-partition. According to preceding lemma there is $y \in (J'')^+ \subseteq (J')^+$ such that $y \cap x_n$ is finite (resp. $|y \cap x_n| \leqq n$, resp. $|y \cap x_n| \leqq 1$).

The next two lemmas help to distinguish the properties of ideals which we investigate.

1.7. Lemma. If an ideal J' is countably generated and not one-generated over an ideal J then J' is not a p^*-ideal.

Proof. Let $\{x_n, n \in \omega\} \subseteq J^+$ be a partition which generates J' over J. If $x \cap x_n$ is finite for every n then $x \notin (J')^*$.

1.8. Lemma. Let $\{x_n; n \in \omega\}$ be a partition of ω into infinite sets, let J_n be an ideal on x_n for every n. Let $x \in J$ if and only if $x \cap x_n \in J_n$ for every n. Then J is an r^*-ideal (resp. p^*-ideal, resp. regular) if and only if every ideal J_n is such too.

Moreover the ideal J is not a p-ideal.

Proof. Let $\{y_n;\, n \in \omega\}$ be a strong J-partition. If each J_n is an r^*-ideal, then there is $x \in J$ such that $|x \cap x_m \cap y_n| \leqq n$ for every $n, m \in \omega$. Put $y = \bigcup_m \bigcup_{n>m} x \cap x_m \cap \cap y_n$. Then $y \in J^*$ and $|y \cap y_n| = |\bigcup_{m<n} x \cap x_m \cap y_n| \leqq n^2$. According to Proposition 1.4 J is an r^*-ideal. Similarly the case of p^*-ideal.

Assume every J_n is regular and $\{y_n,\, n \in \omega\}$ is a partition of ω into finite sets. Construct a decreasing sequence of infinite sets $z_{n+1} \subseteq z_n$ such that $\bigcup\{y_k \cap x_n;\, k \in z_n\} \in J_n$. Let $v \subseteq \omega$ be infinite such that $v - z_n$ is finite for every n. Then $\bigcup\{y_k;\, k \in v\} \in J$. Thus J is regular.

The "only if" direction is easy. Finally, J is not a p-ideal since the partition $\{x_n;\, n \in \omega\}$ is a J-partition and every $x \in J^+$ meets at least one set x_n at an infinite set.

Examples:

1. A dual ideal to the selective ultrafilter is an s & s^*-ideal.
2. The ideal of finite subsets of ω is an s & p^*-ideal and is not regular.
3. An ideal countably and not one-generated over the ideal of finite sets is an s & $\neg p^*$-ideal (Lemmas 1.6 and 1.7).
4. Every maximal r & $\neg s$-ideal is r & $\neg s$ & r^* & $\neg s^*$-ideal.
5. Let x_1, x_2 be a partition of ω into infinite sets, let J_1 be an r & $\neg r^*$ & p^*-ideal on x_1 and let J_2 be an r & $\neg s$ & r^*-ideal on x_2 (e.g. ideals in examples 2 and 4). The ideal J on ω defined by $x \in J$ iff $x \cap x_i \in J_i$ for $i = 1, 2$ is an r & $\neg s$ & p^* & $\neg$r* ideal.
6. Let $\{x_n;\, n \in \omega\}$ be a partition of ω such that $|x_n| > n$. Let an ideal J be generated by selectors of the partition i.e. $x \in J$ if and only if $\exists k\, \forall n |x \cap x_n| \leqq k$. This ideal J is an r & $\neg s$ & $\neg p^*$-ideal.
7. Let $\{a_n,\, n \in \omega\}$ be a decreasing sequence of positive reals, $\lim_{n\to\infty} a_n = 0$ and $\sum_{n \in x} a_n = \infty$.

Define an ideal J on ω: $x \in J$ if and only if $\sum_{n \in x} a_n < \infty$.

Claim. J is a p & p^* & $\neg r$ & $\neg r^*$-ideal and is not regular.

Proof. Since for every countable family of converging series there exists a converging series eventually dominating each of them the ideal J is a p^*-ideal (Proposition 1.2).

Let $\{x_n;\, n \in \omega\}$ be a J-partition. For every n choose a finite set $y_n \subseteq x_n$ such that $\sum_{k \in y_n} a_k \geqq 1$ if $\sum_{k \in x_n} a_k = \infty$ and $\sum_{k \in y_n} a_k \geqq \sum_{k \in x_n} a_k/2$ otherwise. Put $x = \bigcup\{y_n;\, n \in \omega\}$ Then $x \notin J$ and $x \cap x_n$ is finite for every n. Thus J is a p-ideal.

Let $x_n = \{k;\, 1/n^3 \geqq a_k > 1/(n+1)^3\}$. Every set which meets every x_n at n points at most is in J and so J is not an r-ideal. Let $\{x_n;\, n \in \omega\}$ be a partition of ω into finite sets such that $\sum_{k \in x_n} a_k \geqq 1$. Then union or arbitrary subset of this partition is not in J and so J is not regular.

8. An ideal countably and not one-generated over the ideal in example 7 is p & $\neg r$ & $\neg p^*$-ideal (Lemmas 1.6 and 1.8).

9. Let $\{x_n; n \in \omega\}$ be a partition of ω into infinite sets. Put on $\mathscr{P}(x_n)/\text{fin}$ the ordering induced by the reverse inclusion (fin denotes the ideal of finite sets). Let $P = \prod \mathscr{P}(x_n)/ / \text{fin}$ be a product of partially ordered sets with support ω.

Let $G \subseteq P$ be a generic filter over M. Since P is ω-closed M and $M[G]$ have the same countable subsets of M. $G = \prod_{n \in \omega} G_n$. Denote $c_n: \mathscr{P}(x_n) \to \mathscr{P}(x_n)/\text{fin}$, $n \in \omega$, canonical homomorphisms. Since $G_n \subseteq \mathscr{P}(x_n)/\text{fin}$ is a generic filter over M, $J_n = c^{-1}(G_n)$ is a maximal s^*-ideal (see e.g. [5]). Let $x \in J$ if and only if $x \cap x_n \in J_n$ for every $n \in \omega$. Denote c: $\mathscr{P}(\omega) \to P$ the homomorphism: $c(x)(n) = c_n(x \cap x_n)$. Thus $x \in J$ if and only if $c(x) \in G$.

Claim. J. is a $\neg p$ & s^*-ideal.

Proof of Claim. J is not a p-ideal (Lemma 1.7).

We show that J is an s^*-ideal. Let $y_n \in J$, $n \in \omega$, be a collection of disjoint subsets of ω such that the set $y = \bigcup\{y_n; n \in \omega\} \in J^*$. We want to show that there is $z \in J^*$, $z \subseteq y$ such that $|z \cap y_n| \leqq 1$, $n \in \omega$. Since every J_n is an s^*-ideal we may assume that $|y_n \cap x_m| \leqq 1$ and their common element (if exists) we denote $k_{n,m}$. Clearly $p = c(\omega - y) \in G$. It is enough to prove:

$(*)$ $\forall q \leqq p \; \exists q' \leqq q \; \exists z \subseteq \omega \; \forall n |z \cap y_n| \leqq 1 \; \& \; q' \Vdash \check{z} \in J^*$.

If $z \subseteq y$ is arbitrary let $v_m = \{n; k_{n,m} \in x_m \cap z\}$. Thus $z \cap x_m = \{k_{n,m}; n \in v_m\}$. Then $|z \cap y_n| \leqq 1$, $n \in \omega$, if and only if v_m, $m \in \omega$, are mutually disjoint.

If $a \subseteq \omega$ is arbitrary, then $c(a) \Vdash \check{z} \in J^*$ if and only if $x_m - z \subseteq^* x_m \cap a$ for every m ($x \subseteq^* y$ means $x - y$ is finite and $x =^* y$ iff $x \subseteq^* y$ and $y \subseteq^* x$).

So $(*)$ is equivalent to $(**)$:

$(**)$ For arbitrary collection a_m, $m \in \omega$, such that $x_m - y \subseteq^* a_m \subsetneqq^* x_m$, $m \in \omega$, there exist sequences a'_m, $m \in \omega$, and v_m, $m \in \omega$, such that $a_m \subseteq^* a'_m \subsetneqq^* x_m$, $x_m - \{k_{n,m}; n \in v_m\} \subseteq^* a'_m$, $m \in \omega$, and sets v_m, $m \in \omega$, are mutually disjoint.

Proof of $(**)$: Let $x_m - y \subseteq^* a_m \subsetneqq^* x_m$ for every n. Then sets $b_m = \{n; k_{n,m} \in y \cap x_m - a_m\}$, $m \in \omega$, are infinite because $y \cap x_m \subsetneqq^* a_m$. Choose $v_m \subseteq b_m$, $m \in \omega$ infinite mutually disjoint. Put $a'_m = x_m - \{k_{n,m}; n \in v_m\}$, $m \in \omega$. Then $a_m \subseteq a'_m \subsetneqq^* x_m$, $m \in \omega$.

10. Let K be an s^*-ideal. If in Lemma 1.7 $J_n = \{x \subseteq x_n; f_n^{-1}(x) \in K\}$ where f_n: $\omega \to x_n$ is a bijective function, $n \in \omega$, then the ideal J is a $\neg p$ & r^* & $\neg s^*$-ideal.

Proof. The family $y_n = \{f_i(n), i \in \omega\}$, $n \in \omega$, is a strong J-partition. If $x \subseteq \omega$ is such that $|x \cap y_n| \leqq 1$ for every n then $f_0^{-1}(x \cap x_0) \cap f_1^{-1}(x \cap x_1) = \varnothing$ and so at least one set among $f_i^{-1}(x \cap x_i)$, $i = 0, 1$ is not in K^* and so $x \notin J^*$.

11. If ideals J_n, $n \in \omega$, in Lemma 1.7 are p^* & $\neg r^*$-ideals (resp. $\neg p^*$-ideals) then the ideal J is a $\neg p$ & p^* & $\neg r^*$-ideal (resp. $\neg p$ & $\neg p^*$-ideal).

Remark. If CH holds then there are $2^{\aleph_1}$ maximal r-ideals which are not s-ideals as well as $2^{\aleph_1}$ maximal p-ideals which are not r-ideals. This can be proved using Lemma 1.6 in a similar way as it is done for maximal s-ideals in [5] (e.g. by extension of ideals from examples 6, 7).

§ 2. Some characterizations of ideals

In this section we give some equivalent characterizations of the properties of r^*-ideal and regular p^*-ideal (similar to the characterizations of s-ideal and p-ideal given in [5]) which we will need in § 3.

Let T_0 be the set of all finite sequences of finite subsets of ω i.e. $T_0 = {}^{<\omega}([\omega]^{<\omega})$ and let $T_1 = \{s \in T_0;\ \forall n \in \operatorname{dom} s\, |s(n)| \leqq n\}$. The sets T_0, T_1 ordered by inclusion are trees. In the study of ideals we will need some properties of subtrees of $(T_0, \subseteq)$ and $(T_1, \subseteq)$. We write $s * t$ dor the concatenation of two sequences s and t.

2.1. Definition. (i) If A is a tree and $s \in A$ the ramification of A at s is the set $\operatorname{ram}_A(s) = \{a;\ s * (a) \in A\}$ where (a) denotes the sequence $\{(0, a)\}$.

(ii) Let J be an ideal on ω. We say that a tree $A \subseteq T_0$ (resp. $A \subseteq T_1$) is a J-strong tree if for any $s \in A$ there is $x \in J^*$ such that $[x]^{<\omega} \subseteq \operatorname{ram}_A(s)$ (resp. $[x]^{\leqq |s|} \subseteq$ $\subseteq \operatorname{ram}_A(s)$).

(iii) A branch $H\colon \omega \to [\omega]^{<\omega}$ is J-big if $\bigcup \operatorname{rng} H \in J^*$.

2.2 Definition. An ideal J is an r^*-T-ideal (resp. p^*-T-ideal) if every J-strong tree $A \subseteq T_1$ (resp. $A \subseteq T_0$) has a J-big branch.

2.3. Proposition. (i) If J is a p^*-T-ideal (resp. r^*-T-ideal) then J is a p^*-ideal (resp. r^*-ideal).

(ii) If J is a p^*-T-ideal then J is regular.

Proof. (i) Let $\{x_n;\ n \in \omega\}$ be a strong J-partition. We define a J-strong tree $A \subseteq T_0$ (resp. $A \subseteq T_1$) as follows: for $s \in T_0$ (resp. $s \in T_1$) $s \in A$ iff $(\forall k, l, m, n \in \omega)$ $((s(m) \cap$ $\cap\, x_k \neq \emptyset\ \&\ s(n) \cap x_1 \neq \emptyset\ \&\ m < n) \to k < l)$. Let H be a J-big branch of the tree A and $x = \bigcup \operatorname{rng} H$. Then $x \in J^*$ and $x \cap x_n$ is finite (resp. $|x \cap x_n| \leqq n$) for every n.

(ii) Let $\{x_n;\ n \in \omega\}$ be a partition of ω into finite sets. We define a J-strong tree $A \subseteq T_0$ by induction: $\emptyset \in A$ and if $s \in A$, $|s| = n$ and $m_s = \sup\{k;\ x_k \cap$ $\cap \bigcup \operatorname{rng} s \neq \emptyset\}$ then $\operatorname{ram}_A(s) = [\omega - \bigcup\{x_k;\ k \leqq m_s + 1\}]^{<\omega}$. Every branch of the tree A avoids infinitely many sets x_n.

2.4. Definition. An ideal J is r^*-inductive (resp. p^*-inductive) if for every decreasing sequence $\{x_n; n \in \omega\} \subseteq J^*$ there exists $H: \omega \to [\omega]^{<\omega}$ satisfying the following conditions (a)–(d) (resp. (a)–(c)):
(a) if $m < n$ then $\sup H(m) < \sup H(n)$,
(b) $\bigcup \operatorname{rng} H \in J^*$,
(c) $H(0) \subseteq x_0$, $H(n + 1) \subseteq x_{\sup H(n)}$ for $n \in \omega$,
(d) $|H(n)| \leqq n$ for $n \in \omega$.

2.5. Proposition. (i) If J is an r^*-ideal then J is r^*-inductive.
(ii) If J is a regular p^*-ideal then J is p^*-inductive.

Proof. (i) Let $\{x_n; n \in \omega\} \subseteq J^*$ be a decreasing sequence. We show that there is a function H satisfying conditions (a)–(d).

Since r^*-ideal is p^*-ideal by 1.2 there is $x \in J^*$ such that $x - x_n$ is finite for every n and put $r(n) = \sup (x - x_n)$. If r is bounded, then it is obvious how to find the function H. We may assume that r is strictly increasing (if not, then we can take some subsequence of the sequence $\{x_n; n \in \omega\}$). Define $p_0 = 0$, $p_{n+1} = r(p_n)$ for $n \in \omega$.

If $a, b \in x$ and $a \leqq p_n \leqq p_{n+1} < b$ then $b \in x_a$. The sets $u_n = (p_n, p_{n+1}\rangle$ form a partition of ω into finite sets. Since J is regular there is an increasing function h such that $\bigcup\{u_{h(n)}; n \in \omega\} \in J$. Put $v_{n+1} = \{u_k; h(2n) < k < h(2n + 2)\}$ and $v_0 = \omega - \bigcup\{v_n; n \geqq 1\}$. The family v_n, $n \in \omega$, is a strong J-partition. Let $y \in J^*$ be such that $|y \cap v_{n+1}| \leqq n$ (by Proposition 1.4). Put $H(n) = v_{n+1} \cap y$, $n \in \omega$. Then $\bigcup \operatorname{rng} H \in J^*$ and $H(n + 1) \subseteq x_{\sup H(n)}$ because between $H(n)$ and $H(n + 1)$ is an interval u_k.

(ii) The proof is same.

2.6. Lemma. If J is p^*-inductive (resp. r^*-inductive) and $\{x_s; s \in T_0\} \subseteq J^*$ is an arbitrary family then there is a function $H: \omega \to [\omega]^{<\omega}$ such that $\bigcup \operatorname{rng} H \in J^*$, $H(n) \subseteq x_{H \upharpoonright n}$ and $H(n)$ is finite (resp. $|H(n)| \leqq n$) for every n.

Proof. By the finite intersection property we may assume that if $|s| < |t|$ and $\sup \bigcup \operatorname{rng} s \leqq \sup \bigcup \operatorname{rng} t$ then $x_t \subseteq x_s$. Let s_n be the sequence of length $n + 1$ with the constant value $\{n\}$ and put $y_n = x_{s_n}$. Using the hypothesis that J is p^*-inductive (resp. r^*-inductive) there is a function $H: \omega \to [\omega]^{<\omega}$ satisfying conditions (a)–(c) (resp. (a)–(d)) for the decreasing $\{y_n; n \in \omega\}$. By (c) $H(n + 1) \subseteq y_{\sup H(n)}$ and by (a) $y_{\sup H(n)} \subseteq x_{H \upharpoonright (n+1)}$.

2.7. Proposition. If J is p^*-inductive (resp. r^*-inductive) then J is a p^*-T-ideal (resp. r^*-T-ideal).

Proof. Let $A \subseteq T_0$ (resp. $A \subseteq T_1$) be a J-strong tree. If $s \in A$ then we choose some set $x_s \in J^*$ such that $[x_s]^{<\omega} \subseteq \operatorname{ram}_A(s)$ (resp. $[x_s]^{\leqq |s|} \subseteq \operatorname{ram}_A(s)$). If $s \notin A$ we put $x_s = \omega$.

Let H be a function from 2.6. It is not hard to show that H is a branch of the tree A.

Thus we have proved:

2.8. Theorem. (a) The following are equivalent:

(i) J is an r^*-ideal
(ii) J is an r^*-T-ideal
(iii) J is r^*-inductive

(b) The following are equivalent:

(i) J is a regular p^*-ideal
(ii) J is a p^*-T-ideal
(iii) J is p^*-inductive.

§ 3. Generalized Cohen's forcing

In this section we continue the S. Grigorieff's study of generalized Cohen's forcing: if J is an ideal on ω, let $C(J) = \bigcup\{{}^x2;\ x \in J\}$ (i.e. $C(J)$ is the set of two valued functions defined on an element of J) ordered by the reverse inclusion.

We study the properties of generic extensions obtained by this forcing in a dependence on the combinatorial properties of ideals investigated in § 1. The main results of this section are the proofs of theorems A, B.

In the following M is a transitive model of ZFC, $C(J) \in M$ and $G \subseteq C(J)$ is a generic filter over M. In $M[G]$ we define the J-Cohen real g over M: $g(n) = 0$ if and only if $\{(n, 0)\} \in G$. It is clear that $M[G] = M[g]$. Let us remind some notation to be used in the following.

The topology on the Cantor space ${}^\omega 2$ is given by taking sets $[s] = \{f \in {}^\omega 2;\ s \subseteq f\rangle$ for $s \in {}^{<\omega}2$ to be the basic open sets. The production measure μ on ${}^\omega 2$ is determined by declaring that for each $s \in {}^n 2$ $\mu([s]) = 2^{-n}$. We use symbols $\forall^\infty n$, $\exists^\infty n$ as abbreviations for $\exists m\ \forall n > m$, $\forall m\ \exists n > m$. If $f, g \in {}^\omega\omega$ then $f \leqq g$ means $(\forall^\infty n) f(n) \leqq g(n)$, A set A coded in M means that A is Borel with its Borel code in M. $\mathbb{K}$ and $\mathbb{L}$ denote ideals of meager sets and of null sets in the Cantor space respectively. If $M \subseteq N$ are two transitive models and $I = \mathbb{K}$ or $I = \mathbb{L}$ then we define (see [9]):

$\operatorname{cof} I_M^N$ iff for every $A \in I$ coded in N there is $B \in I$ coded in M and $A \subseteq B$
$\operatorname{non} I_M^N$ iff ${}^\omega 2 \cap M$ is not included in a set from I coded in N
d_M^N iff $\forall f \in {}^\omega\omega \cap N\ \exists h \in {}^\omega\omega \cap M\ f \leqq h$
The following characterization of $\operatorname{cof} \mathbb{L}_M^N$ is known (see [1], [7], [10]).

3.1. Proposition. The following are equivalent:

(i) $\operatorname{cof} \mathbb{L}_M^N$
(ii) There exists $f \in {}^\omega\omega \cap M$ such that

$$(\forall h \in {}^{\omega}\omega \cap N)(\exists \varphi \in M)(\forall n \in \omega)(h(n) \in \varphi(n) \ \&\ |\varphi(n)| \leqq f(n))$$

(iii) If $f \in {}^{\omega}\omega \cap M$ is arbitrary increasing function then

$$(\forall h \in {}^{\omega}\omega \cap N)(\exists \varphi \in M)(\forall n \in \omega)(h(n) \in \varphi(n) \ \&\ |\varphi(n)| \leqq f(n))$$

Now we prove Theorem A. We decompose the proof into some parts. The direction (i) $\to$ (ii) is the Proposition 3.3, (ii)$-$(iii) is evident and (iii) $\to$ (i) follows from Proposition 3.5 and Theorem 3.7.

3.2. Lemma. If $p \Vdash \exists m \in \check{\omega}\ h(\check{k}) = m$ and a set $a \subseteq \omega - \operatorname{dom} p$ is finite, then there is a finite set $b \subseteq \omega$ and a condition $q \leqq p$ such that $|b| \leqq |{}^{a}2|$, $a \cap \operatorname{dom} q = \emptyset$ and $q \Vdash h(\check{k}) \in \check{b}$.

Proof. Let ${}^{a}2 = \{r_i; i < t\}$. By induction construct a sequence of conditions $q_i \leqq q_{i-1} \leqq p$ and integers n_i, $i < t$, such that $a \cap q_i = \emptyset$ and $q_i \cup r_i \Vdash h(\check{k}) = \check{n}_i$, Then put $q = q_{t-1}$ and $b = \{n_i; i < t\}$.

3.3. Proposition. If J is an r^*-ideal then cof $\mathbb{L}_M^{M[G]}$ holds,

Proof. According to 3.1 it is enough to prove:

$$(\forall h \in {}^{\omega}\omega \cap M[G])(\exists \varphi \in M)(\forall n \in \omega)(h(n) \in \varphi(n) \ \&\ |\varphi(n)| \leqq 2^{n(n+1)/2}).$$

Let $\underline{h}$ be a name of a function $h \in {}^{\omega}\omega \cap M[G]$ and $p \in G$ is a condition such that $\forall k \in \omega\, p \Vdash \exists m \in \check{\omega}\ \underline{h}(\check{k}) = m$. Let $q \leqq p$ be arbitrary condition. We will construct a J-strong tree $A \subseteq T_1$, a decreasing function $Q: A \to C(J)$ and a function $\psi: A \to \to [\omega]^{<\omega}$ by induction such that for every $s \in A$ $\operatorname{dom} Q(s) \cap \bigcup \operatorname{rng} s = \emptyset$ and if $|s| = k + 2$ then $Q(s) \Vdash$ "$\underline{h}(\check{k}) \in (\phi(s))^{\vee}$" and $|\phi(s) \leqq 2^{k(k+1)/2}$.

Put $\emptyset \in A$, $Q(\emptyset) = q$, $\psi(\emptyset) = \emptyset$. Let $s \in A$, $|s| = k$ and $Q(s)$, $\psi(s)$ are defined. For every $a \in [\omega - \operatorname{dom} Q(s)]^{\leqq k}$ according to 3.2 there exist $q_a \leqq Q(s)$ and $b_a \subseteq \omega$ such that $\operatorname{dom} q_a \cap \bigcup \operatorname{rng}(s * (a)) = \emptyset$, $q_a \Vdash \underline{h}(\check{k}) \in \check{b}_a$ and $|b_a| \leqq 2^{k(k+1)/2}$. Define:

$$s * (a) \in A \quad \text{iff} \quad a \in [\omega - \operatorname{dom} Q(s)]^{\leqq k} \quad \text{and further put}$$

$$Q(s * (a) = q_a \quad \text{and} \quad \psi(s * (a)) = b_a\,.$$

Since J is an r^*-T-ideal (Theorem 2.8) there is a J-big branch H of A. Denote $r = \bigcup\{Q(H \restriction n); n \in \omega\}$. Since r is a function and $\operatorname{dom} r \cap \bigcup \operatorname{rng} H = \emptyset$ then r is a condition which extends q and for every $n \in \omega$ $r \Vdash \underline{h}(\check{n}) \in (\varphi(n))^{\vee}$, where $\varphi(n) = = \psi(H \restriction (n + 1))$ and $|\varphi(n)| \leqq 2^{n(n+1)/2}$.

3.4. Lemma. If J is a p^*-ideal and not an r^*-ideal then $M[G] \models (\forall f \in {}^{\omega}\omega \cap M)\,.\,(\exists h \in {}^{\omega}\omega)(\forall \varphi \in {}^{\omega}\omega \cap M)(\varphi \leqq f \to (\exists^{\infty} n)\, h(n) = \varphi(n))$.

Proof. Let $f \in {}^{\omega}\omega \cap M$ be arbitrary nondecreasing unbounded and let $\{x_n; n \in \omega\}$ be a strong J-partition of ω such that for every $x \subseteq \omega$ if $|x \cap x_n| \leqq f(n)$ for every n then $x \notin J^*$ (Proposition 1.4). Since is a p^*-ideal we can assume that every x_n is finite.

Building in $M[G]$ let $a_n = |\{i \in x_n; g(i) = 1\}|$.

Define: $h(n) = k$ iff $0 \leqq k \leqq f(n)$ and $k \equiv a_n(\mathrm{mod}\,(f(n)+1))$. Let p be arbitrary condition, $m \in \omega$ and $\varphi \in {}^{\omega}\omega \cap M$, $\varphi \leqq f$. There exists $n \geqq m$ such that $|x_n - \mathrm{dom}\, p| > f(n)$ and so there exists a condition $q \leqq p$ defined on the set x_n such that $|\{i \in x_n;\, q(i) = 1\}| \equiv \phi(n)\ (\mathrm{mod}\, f(m) + 1)$ and so $q \models (\exists n > \check{m})\, \underline{h}(n) = (\varphi(n))^{\vee}$. A density argument shows that $(\exists^{\infty} n)\, h(n) = \check{\varphi}(n)$.

3.5. Proposition. If J is a p^*-ideal and not an r^*-ideal then non $\mathbb{L}_M^{M[G]}$ does not hold.

Proof. By the preceding lemma there is $h \in M[G]$ such that $(\forall x \in {}^{\omega}2 \cap M)\,(\exists^{\infty} n)\,.\, h(n) = x \restriction n$. We may assume that $h(n) \in {}^{n}2$ for every n. Then the set ${}^{\omega}2 \cap M \subseteq \bigcup_{n\in\omega} \bigcup_{m>n} [h(m)]$ is null.

3.6. Lemma. If J is not a p^*-ideal then

$$M[G] \models (\exists f \in {}^{\omega}\omega)\,(\exists h \in {}^{\omega}\omega \cap M \text{ unbounded})\,(\exists h' \in {}^{\omega}\omega)\,(\exists^{\infty} n)$$
$$f(n) = h(h'(n))\ \&\ h'(n) \geqq n\,.$$

Proof. We prove an equivalent statement:

$M[G] \models (\exists f \in {}^{\omega}\omega)\,(\forall h \in {}^{\omega}\omega \cap M$ unbounded$)\,(\exists^{\infty} n)\,(\exists m > n)\, f(n) = h(m)$. Let $\{x_n;\, n \in \omega\} \subseteq J$ be a partition of ω such that every subset $x \subseteq \omega$ which has finite intersection with every element of the partition is not in J^*. We may assume that every x_n is infinite. We endow the set ${}^{x_n}2$ with the equivalence relation "to differ finitely many times" and let $\{h_{n,i};\, i \in I\}$ be a set of representatives in this equivalence relation "$\sim$" for every $n \in \omega$. In $M[G]$ define $f \in {}^{\omega}\omega$ as follows: $f(n) = m$ iff $h_{n,i} \sim g \restriction x_n$ for some $i \in I$ and $h_{n,i}$ differs from $g \restriction x_n$ precisely in m elements.

Let $h \in {}^{\omega}\omega \cap M$ be arbitrary unbounded, let p be arbitrary condition and let $k \in \omega$. There exists $n > k$ such that $x_n - \mathrm{dom}\, p$ is infinite. Let $i \in I$ be such that $p \restriction x_n - h_{n,i}$ is finite and let $m \geqq n$ be such that $h(m) \geqq |p \restriction x_n - h_{n,i}|$. Then there is a condition $q \leqq p$ such that $x_n \subseteq \mathrm{dom}\, q$ and $q \restriction x_n$ differs from $h_{n,i}$ precisely in $h(m)$ elements.

A density argument shows that $\exists^{\infty} n\, \exists m \geqq n\ f(n) = h(m)$.

3.7. Theorem. If J is not a p^*-ideal then the union of closed null sets coded in M is a null set in $M[G]$.

Proof. For arbitrary closed null subset $B \in M$ of the Cantor space there exists clopen sets $a_B(n)$ for $n \in \omega$ such that $B = \bigcap_{n\in\omega} a_B(n)$, $\mu(a_B(n)) < 1/n^2$ and $a_B(n) \supseteq a_B(n+1)$. The set C of all clopen sets is countable and if B is not empty then the range of the function a_B is infinite. According to Lemma 3.6 there is $f\colon \omega \to C$ in $M[G]$ such that for every function a_B there exists $h_B \in {}^{\omega}\omega \cap M[G]$ such that the set $x_B = \{n;\, f(n) = a_B(h_B(n))\ \&\ h_B(n) \geqq n\}$ is infinite. For every $n \in x_B$ we have

$\mu(a_B(h_B(n)) \leqq \mu(a_B(n)) < 1/n^2$. We may assume that $\mu(f(n)) < 1/n^2$ for every n. Then the set $A = \bigcap_{n\in\omega} \bigcup_{m>n}(f(m))^*$ contains all closed null sets coded in M and $\mu(A) = 0$ ($(f(m))^*$ denotes a clopen set in $M[G]$ with the same Borel code as $f(m)$).

3.8. Proposition. If J is a regular p^*-ideal then $\mathrm{d}_M^{M[G]}$ holds.

Proof is almost same as the proof of 3.3 where sup $\varphi(n)$ is a function dominating a member of ${}^{\omega}\omega \cap M[G]$.

Let $\{x_n; n \in \omega\} \subseteq J$ is a partition of ω. Define an ideal K on ω as follows:

$$(*) \qquad x \in K \quad \text{if and only if} \quad \bigcup\{x_m; m \in x\} \in J .$$

In [5] is proved in the case J is maximal: there exists a normal function from $C(J)$ into $C(K)$. This remains faithful in the case J is not maximal too. Therefore $M[G]$ contains a K-Cohen real over M.

We recall that an ideal J is not regular if and only if there exists a partition $\{x_n; n \in \omega\}$ of ω into finite sets such that the ideal K defined by $(*)$ is the ideal of finite sets. Since $\mathrm{d}_M^{M[G]}$ implies there is no Cohen real over M in $M[G]$ we have proved:

3.9. Corollary. If J is a p^*-ideal then the following are equivalent:

(i) J is a regular ideal

(ii) There is no Cohen real over M in $M[G]$.

3.10. Proposition. (compare with [2] and [11] If an ideal J is generated by fewer than b sets then J is not regular (b is the minimal cardinality of unbounded family of functions).

Proof. Let $\mathscr{B} \subseteq \mathscr{P}(\omega)$ be a base of an ideal J and $|\mathscr{B}| < b$. For $B \in \mathscr{B}$ let $g_B(n) = \min\{m > n; \langle n, m) - B \neq \emptyset\}$. By definition of b there is $g \in {}^{\omega}\omega$ so that $g_B \leqq g$ for all $B \in \mathscr{B}$. We may assume that g is strictly increasing. Define $a_0 = 0$, $a_{n+1} = g(a_n)$, Put $x_n = \langle a_n, a_{n+1})$ for $n \in \omega$. Let $x \subseteq \omega$ be arbitrary such that $y = \bigcup\{x_n; n \in x\} \in J$. Then y is contained in some set $B \in \mathscr{B}$. But $(\forall^\infty n)\, x_n - B \neq \emptyset$ and so the set x is finite.

3.11. Corollary. If J is an ideal generated by fewer than b sets then there is a Cohen real over M in $M[G]$.

3.12. Lemma. Suppose $p \models$ "$\underset{\sim}{C}$ is nowhere dense in ${}^{\omega}2$", $a \subseteq \omega - \operatorname{dom} p$ is a finite subset and $t \in {}^{<\omega}2$. Then there is $r \in {}^{<\omega}2$ and a condition $q \leqq p$ such that $s \subseteq r$, $a \cap \operatorname{dom} q = \emptyset$ and $q \models [r] \cap \underset{\sim}{C} = \emptyset$.

Proof. Let ${}^{a}2 = \{r_i;\ i < k\}$. Build a sequence of conditions $q_i \leqq q_{i-1} \leqq p$ and a sequence of $t_i \in {}^{<\omega}2$, $t_i \supseteq t_{i-1} \supseteq t$, $i < k$ such that $a \cap \operatorname{dom} q_i = \emptyset$ and $q_i \cup r_i \Vdash [t_i] \cap \underset{\sim}{C} = \emptyset$. Now let $q = q_{k-1}$ and $x = t_{k-1}$.

3.13. Proposition. If J is a regular p^*-ideal then cof $\mathbb{K}_M^{M[G]}$ holds.

Proof. Let $\{t_n;\ n \in \omega\}$ be an enumeration of ${}^{<\omega}2$. Assume $p \Vdash$ "C is nowhere dense" and $p \in G$. Let $q \leqq p$ be arbitrary. We will construct a J-strong tree $A \subseteq T_0$, a decreasing function $Q: A \to C(J)$ and a function $\varphi: A \to {}^{<\omega}2$ such that $\operatorname{dom} Q(s) \cap \bigcup \operatorname{rng} s = \emptyset$ for every $s \in A$ and if $|s| = k + 1$ then $t_k \subseteq \varphi(s)$ and $Q(s) \Vdash [(\varphi(s))^{\vee}] \cap \underset{\sim}{C} = \emptyset$.

Put $\emptyset \in A$, $Q(\emptyset) = q$, $\varphi(\emptyset) = \emptyset$. Let $s \in A$, $|s| = k$ and $Q(s)$, $\varphi(s)$ are defined. For every $a \in [\omega - \operatorname{dom} Q(s)]^{<\omega}$ according to 3.12 there exist $q_a \leqq Q(s)$ and $r_a \supseteq t_k$ such that $\operatorname{dom} q_a \cap \bigcup \operatorname{rng} s * (a) = \emptyset$ and $q_a \Vdash [\check{r}_a] \cap \underset{\sim}{C} = \emptyset$. Define: $s * (a) \in A$ iff $a \in [\omega - \operatorname{dom} Q(s)]^{<\omega}$ and further put $Q(s * (a)) = q_a$ and $\varphi(s * (a)) = r_a$.

According to Theorem 2.8 (b) there is a J-big branch H in A. Let r be a condition extending $Q(H \upharpoonright n)$ for $n \in \omega$ (it is possible since $\operatorname{dom} (\bigcup_n Q(H \upharpoonright n)) \cap \bigcup \operatorname{rng} H = \emptyset$). Then $r \Vdash$ "the set $D = \bigcup \{[(\varphi(H \upharpoonright (n + 1)))^{\vee}];\ n \in \check{\omega}\}$ is open dense coded in M and $D \cap \underset{\sim}{C} = \emptyset$". A density argument shows that every nowhere dense set in $M[G]$ is covered by a nowhere dense set coded in M and so (see [8]) every meager set in $M[G]$ is covered by a meager set coded in M.

Proof of Theorem B. The direction (i) → (iii) is Proposition 3.13.

(iii) → (ii) follows from the Cichoń's diagram (see [9]).

(ii) → (i): If J is a p^*-ideal and it is not regular then $\mathrm{d}_M^{M[G]}$ does not hold by 3.9. If J is not p^*-ideal then the function f in 3.6 is dominated by no function from M.

3.14. Proposition (see [5]). If J is a regular p^*-ideal then for arbitrary ordinal α if $\mathrm{cf}^M(\alpha) > \omega$ then $\mathrm{cf}^{M[G]}(\alpha) > \omega$.

Proof is similar to the proof of 3.3. Instead of a function $h \in {}^{\omega}\omega \cap M[G]$ consider arbitrary function $h: \omega \to \alpha$. Then $\bigcup \{\varphi(n);\ n \in \omega\}$ will be a countable set of ordinals covering rng h.

3.15. Proposition (see [5]). If J is not p^*-ideal then $M[G] \Vdash \mathrm{cf}((2^{\omega})^M)) = \omega$.

Proof. Let $\{x_n;\ n \in \omega\} \subseteq J$ be a partition of ω such that if $x \cap x_n$ is finite for each $n \in \omega$ then $x \notin J^*$. We may assume that each x_n is infinite. Let $\lambda = (2^{\omega})^M$ and let ${}^{x_n}2 = \{p_{n,\alpha};\ \alpha \in \lambda\}$ for $n \in \omega$. In $M[G]$ define a function $f: \omega \to \lambda$ as follows:

$$f(n) = \alpha \quad \text{if and only if} \quad g \upharpoonright x_n = p_{n,\alpha}\,.$$

Let p be a condition. There is n such that $x_n - \operatorname{dom} p$ is infinite. Then given arbitrary $\alpha \in \lambda$ there exists $\beta > \alpha$ such that p and $p_{n,\beta}$ are compatible. Put $q = p \cup p_{n,\beta}$ then $q \Vdash \underset{\sim}{f}(\check{n}) > \check{\alpha}$. A density argument shows that f is cofinal with λ.

3.16. Corollary. Assume J is regular. Then the following are equivalent:

(i) J is a p^*-ideal.
(ii) $M[G] \Vdash \mathrm{cf}((2^\omega)^M) > \omega$.
(iii) The union of closed null sets coded in M is not a null set in $M[G]$.

Proof. (i) $\to$ (ii) by 3.14 and 3.15. The direction (iii) $\to$ (i) is Theorem 3.7 and (i) $\to$ (iii) follows from Theorem B since $\mathrm{d}_M^{M[G]}$ implies (iii) (see [7]).

3.17. Proposition. Let κ be an infinite cardinal number and J be κ-generated and not one-generated (over the ideal of finite sets). Then $M[G] \models |(2^\omega)^M| = |\kappa|$. Moreover if $\kappa = \omega$ then r.o. $(C(J)) = Col(\omega, 2^\omega)$.

Proof. Let $\mathscr{B} \subseteq J$ be a base of J and $|\mathscr{B}| = \kappa$. Thus $x \in J$ iff $x \subseteq y$ for some $y \in \mathscr{B}$. Denote $\mathscr{B}' = \{x - y;\ x, y \in \mathscr{B}\ \&\ x - y \text{ is infinite}\}$. Let $\lambda = (2^\omega)^M$ and ${}^x2 = \{p_{x,\alpha};\ \alpha \in \lambda\}$ for $x \in \mathscr{B}'$. In $M[G]$ define a function $f\colon \mathscr{B}' \to \lambda$ as follows: $f(x) = \alpha$ if and only if $g \upharpoonright x = p_{x,\alpha}$. As J is not one-generated for every condition p there is $x \in \mathscr{B}'$ such that $\mathrm{dom}\, p \cap x = \emptyset$. Thus for every $\alpha \in \lambda$ and p we can extend p to some condition q such that $q \Vdash \check{\alpha} \in \mathrm{rng} f$. Therefore $\lambda = \mathrm{rng} f$.

In the case $\kappa = \omega$ see [6], Lemma 25.11.

§ 4. Generalized Silver's forcing

If J is an ideal on ω, let $S(J) = \bigcup\{{}^x2;\ x \in \mathscr{P}(\omega) - J^*\}$ ordered by the reverse inclusion. Let $G \subseteq S(J)$ be a generic filter over M. The J-Silver real is defined as follows: $g(n) = 0$ if and only if $\{(n, 0)\} \in G$. Clearly $M[G] = M[g]$.

4.1. Definition (see [5]). An ideal J on ω is c.d.s. if for every decreasing sequence $\{x_n, n \in \omega\}$ of sets from J^+ there is a set $x \in J^+$ such that $x - x_n \in J$ for every n.

Let $\mathscr{P}(\omega)/J$ be a quotient algebra. We put on it the ordering induced by the reverse inclusion in $\mathscr{P}(\omega)$. Let $c\colon \mathscr{P}(\omega) \to \mathscr{P}(\omega)/J$ be the canonical homomorphism. If $G_1 \subseteq \mathscr{P}(\omega)/J$ is a generic filter over M then $c^{-1}(G_1)$ is a base of an ideal on ω in $M[G_1]$. Denote J^0 the ideal generated by $c^{-1}(G_1)$. Clearly $J \subseteq J^\circ$ and $M[G_1] = M[J^0]$. If J is c.d.s. (i.e. $\mathscr{P}(\omega)/J$ is ω-closed) then M and $M[G_1]$ have the same countable subsets of M and J^0 is a maximal ideal on ω extending J.

The function T from $S(J)$ into $\mathscr{P}(\omega)/J$ defined by $T(p) = c(\mathrm{dom}\, p)$ is a normal function (see [5]). Therefore if $G \subseteq S(J)$ is a generic filter over M then $G_1 = T(G)$ is a generic subset of $\mathscr{P}(\omega)/J$ over M and G is a generic subset of $T^{-1}(G_1)$ over $M[G_1]$.

Conversely. If $G_1 \subseteq \mathscr{P}(\omega)/J$ is a generic filter over M and $G \subseteq T^{-1}(G_1)$ is a generic filter over $M[G_1]$ then G is a generic subset of $S(J)$ over M. Since $T^{-1}(G_1)$ is a dense subset of $C(J^0)$ in $M[G_1]$ we have:

4.2. Proposition (see [5]). A J-Silver real over M coincides with a J^0-Cohen real over $M[J^0]$.

4.3. Proposition. If J is a p-ideal (resp. r-ideal) then J^0 is a maximal p-ideal (resp. r-ideal) in $M[J^0]$.

Proof. Since p-ideal is c.d.s. J^0 is a maximal ideal. Let $\{x_n; n \in \omega\}$ be a J^0-partition of ω in $M[J^0]$. The set $\{c(x_n); n \in \omega\}$ is a subset of G and lies in M. Therefore there is a set $x \subseteq \omega$ such that $c(x) \leqq c(x_n)$ for every n and $c(x) \in G_1$. Let $c(y) \leqq c(x)$ be arbitrary condition and let J' be the ideal generated over J by the set y. The ideal J' is a p-ideal (Lemma 1.5) and the given partition is a J'-partition. Choose $z \in (J')^+$ such that $z \cap x_n$ is finite for every n. As $z \notin J'$ then $z - y \notin J$ and so $(\omega - z) \cup y \notin J^*$. Therefore $r = c((\omega - z) \cup y)$ is a condition, $r \leqq c(y)$ and $r \Vdash z \notin J^0$.

Similarly in the case J is an r-ideal.

4.4. Theorem. If J is c.d.s. then there exist $a, b \in \text{r.o.}\ (S(J))$ such that

(a) $a = [\![\text{cof}\ \mathbb{K}_M^{M[G]}]\!] = [\![\text{d}_M^{M[G]}]\!] = [\![\text{cf}\,(2^\omega)^\vee > \omega]\!] = [\![$the union of closed null sets coded in M is not null$]\!] \leqq [\![\omega_1 = \check{\omega}_1]\!]$

(b) $b = [\![\text{cof}\ \mathbb{L}_M^{M[G]}]\!] = [\![\text{non}\ \mathbb{L}_M^{M[G]}]\!]$

(c) $b \leqq a$

(d) J is a p-ideal if and only if $a = 1$

(e) J is an r-ideal if and only if $b = 1$.

Proof. Since $T: S(J) \to \mathscr{P}(\omega)/J$ is normal so r.o. $(\mathscr{P}(\omega)/J)$ is isomorphic with some complete subalgebra of r.o. $(S(J))$. Let e: r.o. $(\mathscr{P}(\omega)/J) \to$ r.o. $(S(J))$ be a complete embedding. Put $a = e([\![J^0$ is a p^*-ideal in $M[J^0]]\!])$, $b = e([\![J^0$ is an r^*-ideal in $M[J^0]]\!])$. Then (a), (b) follows from Theorems A, B, 3.14, 3.16 and 4.2 since forcing with $\mathscr{P}(\omega)/J$ adds no reals and every maximal ideal is regular. (c) is clear.

The "only if" direction of the statements (d), (e) follow from 4.3.

Conversely. Let J be not a p-ideal. There is a J-partition $\{x_n; n \in \omega\}$ such that if $x \cap x_n$ is finite for every n then $x \in J$. We may assume that $x_n \in J$ for every $n \geqq 1$. (Since J is c.d.s. there is $x \in \mathscr{P}(\omega) - J$ such that $x_n - x \in J$ for every n. Take $x'_0 = x_0 \cup x, x'_n = x_n - x,\ n \geqq 1$). Since $c(x_0) \Vdash x_0 \in \underline{J}^0$ then $c(x_0) \Vdash$ "$\underline{J}^0$ is not a p^*-ideal". Similarly (e).

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