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Acta Universitatis Carolinae. Mathematica et Physica, Vol. 31 (1990), No. 2, 59--63

Persistent URL: http://dml.cz/dmlcz/701954

Orthogonal Partitions

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Received 11 March 1990

In this paper we introduce and study a notion of orthogonal partitions of ω which is in a certain sense dual to the notion of almost disjoint subsets of ω. We consider maximal families of pairwise orthogonal partitions and dual matrices.

0. Notation

We shall use notation from [1]. Let us recall it. Let $(\omega)^\omega$ be the set of all infinite partitions of ω. For $X, Y \in (\omega)^\omega$ $X \leqq Y$ means that X is coarser than Y (or equivalently Y is finer than X), i.e. each block of Y is a subset of some block of X. Let $(X)^\omega$ be the set of all infinite partitions coarser than X. For $X \in (\omega)^\omega$ and $n \in \omega$ we write $X[n] = \{x \cap n: x \in X\} \setminus \{\emptyset\}$. Here $n = \{0, 1, \ldots, n - 1\}$ and so $X[n]$ is a partition of n. It is called a segment. We write $s < {}^*X$ to mean that s is a segment of X, i.e. $s = X[n]$ for some $n \in \omega$. Then we also write $lh(s) = n$ and $|s|$ = the number of blocks in s.

Let s, t be segments. We write $s < {}^*t$ to mean that $lh(s) < lh(t)$ and $s = t[lh(s)]$. For any sequence (s_n) of segments such, that for every $n \in \omega$ $s_n < {}^*s_{n+1}$ let $\lim_{n\in\omega} s_n =$ = the unique $Y \in (\omega)^\omega$ such, that for every $n \in \omega$ $s_n < {}^*Y$. We write $s \leqq {}^*t$ to mean that $s < {}^*t$ or $s = t$. We write $s \leqq t$ to mean that $lh(s) = lh(t)$ and s is coarser than t. Finally we write $s \leqq X$ to mean that $s \leqq X[lh(s)]$. For $X \in (\omega)^\omega$ and $s \leqq X$ let $(s, X) = \{Y \in (\omega)^\omega: s < {}^*Y \leqq X\}$. We call the set a dual Ellentuck neighborhood. The dual Ellentuck topology on $(\omega)^\omega$ is the topology whose basic open sets are the dual Ellentuck neighborhoods.

1. Orthogonal partitions

Definition 1. We say that infinite partitions X, Y are *orthogonal* if there is no infinite partition which is coarser than both X and Y, i.e. $(X)^\omega \cap (Y)^\omega = \emptyset$.

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Examples. Partitions X and Y below are orthogonal
$1°$ $X = \{\{1, 2\}, \{3, 4\}, \{5, 6\}, \ldots\}$; $Y = \{\{1, 2, 3\}, \{4, 5\}, \{6, 7\}, \ldots\}$
$2°$ $X = \{2N\} \cup \{\{n\}: n \in 2N + 1\}$; $Y = \{\{n\}: n \in 2N\} \cup \{2N + 1\}$

Proposition 1. *There is a family of 2^ω pairwise orthogonal partitions.*

Proof. Let X be an arbitrary partition of ω into ω infinite blocks, say $X = \{x_i: i \in \omega\}$; $x_i = \{n_{ik}: k \in \omega\}$ for $i \in \omega$.

For every function $f \in {}^\omega 2$ different from the function χ everywhere equal 1 we define a partition

$$X_f = \{n_{ik}: f(k) = 1,\ k \in \omega,\ i \in \omega\} \cup \{x_i \setminus \{n_{ik}: f(k) = 1,\ k \in \omega\}: i \in \omega\}\ .$$

It is obvious that for different $f, g \in {}^\omega 2 \setminus \{\chi\}$ X_f and X_g are orthogonal. ■

Consider maximal families of pairwise orthogonal partitions, i.e. such a family $\mathscr{R}$ of pairwise orthogonal partitions that $|\mathscr{R}| \geqq 2$ and for every infinite partition X there is some partition $Z \in \mathscr{R}$ such, that $(X)^\omega \cap (Z)^\omega \neq \emptyset$.

Theorem 1. *If $\mathscr{R}$ is a maximal family of pairwise orthogonal partitions, then $|\mathscr{R}| \geqq \omega_1$.*

The proof will be given in a few lemmas.

Lemma 1. *For every finite family of pairwise orthogonal partitions there is a partition orthogonal to each member of the family.*

Proof. Let $\mathscr{R} = \{X_i: i = 1, 2, \ldots, n\}$ be a family of n pairwise orthogonal partitions. For every $i = 1, 2, \ldots, n$ the set $\bigcup\{x: x \in X\ \&\ |x| \geqq 2\}$ is infinite and one of the following cases holds:

case $1°$ X_i has an infinite block;

case $2°$ Every block of X_i is finite, but there are infinitely many blocks having at least two elements.

We may safely assume that the *case* $1°$ holds for first k partitions, $k \leqq n$, namely $X_1, X_2, \ldots, X_k$ and the *case* $2°$ holds for next $n - k$ partitions, namely $X_{k+1}, X_{k+2}, \ldots, X_n$.

For $i = 1, 2, \ldots, k$ let $A_i = \{a_{ij}: j \in \omega\}$ be an arbitrary infinite block of X_i. For $i = k + 1, k + 2, \ldots, n$ let $B_i = \{b_{ij}: j \in \omega\}$ be a family of all at least two-element blocks of X_i and $b_{ij} = b^0_{ij} \cup b^1_{ij}$ be an arbitrary partition of b_{ij} into two non-empty sets, for $j \in \omega$. Now construct a partition $X = \{x_j: j \in \omega\} \cup \{y\}$ as follows.

Assume inductively, that we have already constructed blocks $x_0, x_1, \ldots, x_m$. For $i = 1, 2, \ldots, k$ let

$$i(m + 1) = \min\{j: a_{ij} \notin \bigcup\{x_1: l = 1, 2, \ldots, m\} \cup \{a_{ll(m+1)}: l = 1, 2, \ldots, i - 1\}\}\ .$$

For $i = k + 1,\ k + 2, \ldots, n$ let $i(m + 1) = \min\{j: b_{ij} \cap (\bigcup\{x_l: l = 1, 2, \ldots, m\} \cup \{a_{ll(m+1)}: l = 1, 2, \ldots, k\} \cup \bigcup\{b_{ll(m+1)}: l = 1, 2, \ldots, i - 1\} = \emptyset\}$. Let

$$x_{m+1} = \{a_{ii(m+1)}: i = 1, 2, \ldots, k\} \cup \bigcup\{b^1_{ii(m)} b^0_{ii(m+1)}: i = k + 1, k + 2, \ldots, n\}\ .$$

Having defined all x_m, for $m \in \omega$, define $y = \omega \setminus \bigcup\{x_m: m \in \omega\}$. It is easy to see, that the partition X is orthogonal to each X_i, for $i = 1, 2, \ldots, n$. ■

For any segments s, t and the block a of s such that $0 \in a \in s$ let $s \hat{} t = s \setminus \{a\} \cup \cup \{x \in t: x \cap lh(s) = \emptyset\} \cup \{a \cup \bigcup\{x \setminus lh(s): x \in t \,\&\, x \cap lh(s) \neq \emptyset\}\}$.

It is obvious, that for any s, t $s \leqq {}^*s \hat{} t$ and $lh(s \hat{} t) = \max(lh(s), lh(t))$.

Lemma 2. *For any orthogonal partitions X, Y the following holds $\forall s \leqq X\ \exists t \leqq Y$ $(|s \hat{} t| = |s| + 1 \,\&\, \forall u(u \leqq s \hat{} t \,\&\, u \leqq X \Rightarrow |u| \leqq |s|))$.*

Proof. Let $v < {}^*Y$ be such a segment, that $|s \hat{} v| = |s| + 1$ and let $y_0, y_1, \ldots, y_m$ be segments of Y defined by v. Since $|s \hat{} t| = |s| + 1$ we can assume, that $y_i \cap lh(s) \neq \emptyset$, for $i = 0, 1, \ldots, m - 1$, and $y_m \cap lh(s) = \emptyset$. Since X, Y are orthogonal there are infinitely many triples $(z_0, z_1, z) \in Y \times Y \times X$ such that $z_0 \neq z_1$ and $z_0 \cap z \neq \emptyset \neq z \cap z_1$. Take such a triple with additional property $lh(v) \cap z_0 = \emptyset = lh(v) \cap z_1$.

First define a partition $Y' \leqq Y$

$$Y' = \{y_0 \cup z_0, y_1, \ldots, y_{m-1}, y_m \cup z\} \cup Y \setminus \{y_0, y_1, \ldots, y_m, z_0, z_1\}.$$

Let $n = \max(\min z_0 \cap z, \min z_1 \cap z)$. Taking $t = Y'[n]$ we are done. ■

Similarly we prove the following generalization of the Lemma 2.

Lemma 3. *For any orthogonal partitions $X_1, X_2, \ldots, X_n$, Y, and for every $s_1 \leqq X_1$, $s_2 \leqq X_2, \ldots, s_n \leqq X_n$,*

if $(\forall i \leqq n - 1)(\forall u)(u \leqq s_1 \hat{} s_2 \hat{} \ldots \hat{} s_n \,\&\, u \leqq X_i \Rightarrow |u| \leqq |s_1 \hat{} s_2 \hat{} \ldots \hat{} s_i|)$

then there exists $t \leqq Y$ with $|s_1 \hat{} s_2 \hat{} \ldots \hat{} s_n \hat{} t| = |s_1 \hat{} s_2 \hat{} \ldots \hat{} s_n| + 1$ and such that

$$(\forall i \leqq n)(\forall u)(u \leqq s_1 \hat{} s_2 \hat{} \ldots \hat{} s_n \hat{} t \,\&\, u \leqq X_i \Rightarrow |u| \leqq |s_1 \hat{} s_2 \hat{} \ldots \hat{} s_i|).$$

Lemma 4. *For any countable family of pairwise orthogonal partitions there is a partition orthogonal ot each member of the family.*

Proof. Let $\{X_i: i \in \omega\}$ be a family of pairwise orthogonal partitions. Let $s_0 < {}^*X_0$ be arbitrary. Define segments $s_i \leqq X_i$, for $i \in \omega$, inductively as in Lemma 3.

The partition $X = \lim\limits_{n \in \omega}{}^{\cdot}(s_0 \hat{} s_1 \hat{} \ldots \hat{} s_n)$ will work. ■

Finally we will see that under MA every maximal family of pairwise orthogonal partitions has power continuum.

For any family $\mathscr{R}$ of pairwise orthogonal partitions such that $|\mathscr{R}| < 2^\omega$ let $\boldsymbol{P}_{\mathscr{R}} = \{(s, F): s\text{-segment}, F \subseteqq \mathscr{R} \,\&\, |F| < \omega\}$. We say, that a condition (s, F) is stronger than a condition (t, H) if

(i) $s^* \geqq t \,\&\, F \supseteq H$;

(ii) $\forall X \in H\ \forall r\text{-segment}\ (r \leqq s \,\&\, r \leqq X \Rightarrow |r| \leqq |t|)$.

It is easy to see, that $s \neq t$ for any incompatible (s, F) and (t, H). Hence

Proposition 2. $P_{\mathscr{R}}$ satisfies c.c.c.

Definition 2. For any filter G in $P_{\mathscr{R}}$ let $X_G = \lim_{(s,F)\in G} s$.

The following is an easy consequence of the definitions.

Proposition 3. *Let G be a filter in $P_{\mathscr{R}}$. Then for any $(s, F) \in G$ and any $X \in F$ we have $(\forall r\text{-segment})\ (r \leqq X_G \ \&\ r \leqq X \Rightarrow |r| \leqq |s|)$.*

Proposition 4. *For any $n \in \omega$ and $X \in \mathscr{R}$ the sets $A_n = \{(s, F): |s| \geqq n\}$ and $B_X = \{(s, F): X \in F\}$ are dense in $P_{\mathscr{R}}$.*

Proof. Density of B_X is obvious. To prove density of A_n one can use operation '^'. ∎

Theorem 2. *MA implies, that every maximal family of pairwise orthogonal partitions has power 2^{ω}.*

2. Dual matrices

Definition 3. A family of maximal families of pairwise orthogonal partitions is called *a dual matrix*. A dual matrix $\mathscr{B}$ is called *shattering* if for any infinite partition X there are a family $\mathscr{R} \in \mathscr{B}$ and partitions $X_1, X_2 \in \mathscr{R}$ such, that $X_1 \neq X_2$ and $(X_1)^{\omega} \cap (X)^{\omega} \neq \emptyset \neq (X_2)^{\omega} \cap (X)^{\omega}$. (Then we say that $\mathscr{B}$ and $\mathscr{R}$ *shatter* X).

Definition 4. $\lambda = \min\{|\mathscr{B}|: \mathscr{B}$ is a dual shattering matrix$\}$

Theorem 3. $\omega_1 \leqq \lambda \leqq 2^{\omega}$.

Proof. The inequality $\lambda \leqq 2^{\omega}$ is obvious. Let us prove $\omega_1 \leqq \lambda$.

For segments s, t let

$$s^*t = \{x \cap y: x \in s,\ y \in t\} \cup \{x \setminus lh(t): x \in s\} \cup \{y \setminus lh(s): y \in t\} \setminus \{\emptyset\}\,.$$

Obviously $lh(s^*t) = \max(lh(s), lh(t))$ and $|s^*t| \geqq \max(|s|, |t|)$.

Let $\mathscr{B} = \{\mathscr{R}_i: i \in \omega\}$ be an arbitrary countable matrix. Using the operation '*' we will construct a partition X which is not shattered by $\mathscr{B}$.

Let $s_0 < {}^*X_0 \in \mathscr{R}_0$ be arbitrary. Assume inductively that we have already constructed sequences $s_i < {}^*X_i \leqq Y_i \in \mathscr{R}_i$, for $i = 0, 1, \ldots, n$. Since $\mathscr{R}_{n+1}$ is maximal there is some $Y_{n+1} \in \mathscr{R}_{n+1}$ such that $(X_n)^{\omega} \cap (Y_{n+1})^{\omega} \neq \emptyset$. Let X_{n+1} be an arbitrary element of that intersection and $s_{n+1} < {}^*X_{n+1}$ such, that $|s_{n+1}| = |s_n| + 1$. Let $X = \lim_{n\in\omega} s_0 * s_1 * \ldots * s_n$. From construction follows, that for any $n \in \omega$ and $j \geqq n$ $s_j \leqq X_n$. Thus for any $n \in \omega$ $(s_n, X) \subseteqq (X_n)^{\omega}$, so X cannot be shattered by any $\mathscr{R}_i \in \mathscr{B}$. ∎

Lemma 5. *Let $\mathscr{B}$ be a dual matrix of power less than λ. Then there is a maximal family $\mathscr{R}$ of pairwise orthogonal partitions such, that*

(i) $\forall X \in \mathscr{R}\, \forall \mathscr{R}' \in \mathscr{B}\, \exists X' \in \mathscr{R}'\, \exists s(s \leqq X \ \&\ s \leqq X' \ \&\ (s, X) \subseteqq (s, X'))$;

(ii) $\forall X \in (\omega)^{\omega}$ *$(X$ is shattered by $\mathscr{B} \Rightarrow X$ is shattered by $\mathscr{R})$.*

Proof. The set of all infinite partitions which are not shattered by $\mathscr{B}$ is open and dense in the dual Ellentuck topology. Thus we can construct a maximal family of pairwise orthogonal partitions from elements of the set. Such a family obviously satisfies (i), (ii). ∎

As an easy consequence of the above lemma we obtain

Theorem 4. *λ is a regular cardinal.*

Proposition 5. Con (ZFC + $\lambda < 2^\omega$).

Proof. Let $M_0 \subsetneqq M_1 \subsetneqq \ldots \subsetneqq M_\alpha \subsetneqq \ldots \subsetneqq M$ be models of ZFC such, that $M_0 \vdash$ CH, $M_1 \vdash$ MA + $2^\omega = \omega_1, \ldots, M_\alpha \vdash$ MA + $2^\omega = \omega_\alpha, \ldots, (\alpha < \omega_1)$ and $M \vdash$ $\vdash 2^\omega = \omega_{\omega_1}$. Then $M \vdash \lambda < 2^\omega$. ∎

Proposition 6. *There is a family of λ nowhere dense sets in the dual Ellentuck topology which covers the set of all infinite partitions of ω.*

Proof. It is simple reformulation of the analogous proposition from [2].

Remark. All results of this paragraph were inspired by analogous results from [2] and [3].

References

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[3] Balcar B., Pelant J., Simon P., The space of ultrafilters on N covered by nowhere dense sets, Fund. Math. 110, 11—24 (1980).