Jacek Cichoń

On dense subsets of rational numbers

Acta Universitatis Carolinae. Mathematica et Physica, Vol. 42 (2001), No. 2, 5--10

Persistent URL: http://dml.cz/dmlcz/702071

On Dense Subsets of Rational Numbers

JACEK CICHOŃ

Wrocław

Received 11. March 2001

In this article we consider the family of dense subsets of rational numbers as a partially ordered set. We define cardinal numbers $\mathbf{p}_{\mathbb{Q}}$ and $\mathbf{t}_{\mathbb{Q}}$ for this partial order and we prove that $\mathbf{p}_{\mathbb{Q}} = \mathbf{p}$ and $\mathbf{t}_{\mathbb{Q}} = \mathbf{t}$, where $\mathbf{p}$ and $\mathbf{t}$ are the classical cardinal numbers describing combinatorial properties of the family of all infinite subsets of natural numbers. We also consider some variant of spli ting number of dense subsets of rationals.

1. Introduction

The structure of the family $P(\omega)/fin$ of all infinite subsets of natural nuimbers ω modulo finite sets has been intensively investigated by many authors. One reason for its importance is a natural connection of this structure with the Čech-Stone compactification of natural numbers. Some results are contained in the van Douwen diagram (see [**4**]) which illustrates inequalities between certain cardinal numbers connected with properties of the structure $P(\omega)/fin$. In this paper we shall present some results about the family of dense subsets of rational numbers $\mathbb{Q}$ modulo finite sets.

We use the standard set theoretical notations. The set of natural numbers is denoted by ω. We identify the set ω with the first infinite cardinal number. The cardinality of a set A is denoted by $|A|$. The family of all functions from set A into set B is denoted by B^A. Cardinal numbers are usually denoted by small Greek letters. If κ is a cardinal number then $[A]^\kappa$ denotes the set $\{X \subseteq A : |X| = \kappa\}$. If A, B are sets then $A \subseteq^* B$ means that $|A \setminus B| < \omega$. A family $X \subseteq [\omega]^\omega$ is *centered* if $|\bigcap F_0 = \omega$ for each finite $F_0 \subseteq X$. By $\mathbf{p}$ we denote the least cardinality of

Institute of Mathematics, University of Wrocław, Pl. Grunwaldski 2/4, 50-384 Wrocław, Poland

1991 *Mathematics Subject Classification.* 03E05, 03E35.
Key wo ds and phrases. dense sets, rational numbe s, tower number, splitting number, reaping number.

a centered family $X \subseteq [\omega]^\omega$ such that there is no infinite set $d \in [\omega]^\omega$ such that $d \subseteq^* x$ for each $x \in X$. By $\mathbf{t}$ we denote the least cardinal number κ such that there exists a $\subseteq^*$-decreasing function $f : \kappa \to [\omega]^\omega$ such that there is no infinite set $d \in [\omega]^\omega$ such that $d \subseteq^* f(\alpha)$ for each $\alpha < \kappa$.

If $\varphi, \psi \in \omega^\omega$ then the relation $\varphi \leq^* \psi$ denotes that $(\exists n \in \omega)(\forall k > n)(\varphi(n) \leq \psi(n))$. A family $F \subseteq \omega^\omega$ is *unbounded* if there is no $\psi \in \omega^\omega$ such that $\varphi \leq^* \psi$ for each $\varphi \in F$. By $\mathbf{b}$ we denote the least cardinality of an unbounded family of function from ω^ω. It is well known that $\omega_1 \leq \mathbf{p} \leq \mathbf{t} \leq \mathbf{b} \leq 2^\omega$. By d we denote the least cardinality of a such family D of subsets of ω that $(\forall f \in \omega^\omega)(\exists d \in D)(f \leq^* d)$ (see [**1**]).

The family of all dense subsets of $\mathbb{Q}$ is denoted by $D(\mathbb{Q})$. The set of real numbers is denoted by $\mathbb{R}$. Let $\mathbb{L}$ denotes the σ-ideal of Lebesgue measure zero subsets of $\mathbb{R}$ and let $\mathbb{K}$ denotes the σ-ideal of first category subsets of $\mathbb{R}$. If I is an arbitrary ideal of sets then let $add(I) = \min\{|S| : S \subseteq I \wedge \bigcup S \notin I\}$ and $cov(I) = \min\{S \subseteq I \wedge \bigcup S = \bigcup I\}$.

2. Main result

We begin our consideration with one classical observation formulated by W. Sierpiński (see e.g. [**3**]).

Lemma 1. *(Sierpiński) There exists a function $F : \mathbb{Q} \to \omega$ such that*
1. $(\forall n \in \omega)(|F^{-1}(\{n\})| < \omega)$
2. $(\forall A \in [\omega]^\omega)(F^{-1}(A)$ *is a dense subset of* $\mathbb{Q})$.

Proof. Let $(p_n)_{n\in\omega}$ be an enumeration of the set of all prime numbers. Let $f = \{\langle \frac{i}{p_n}, n\rangle : n \in \omega \wedge 0 < i < p_n\}$ and $D = dom(f)$. Then D is a dense subset of the interval $(0, 1)$ without end-points. Let $\varphi : \mathbb{Q} \to D$ be an order isomorphism between structures $(\mathbb{Q}, <)$ and $(D, <)$. Then $F = f \circ \varphi$ is a required function. ∎

W. Sierpiński used the above Lemma for construction of a family of dense almost disjoint subsets of rational numbers. Namely, suppose that A is a family of pairwise (almost) disjoint subsets of natural numbers and let F be a function from Lemma 1. Then $\{F^{-1}(X) : X \in A\}$ is a family of dense pairwise (almost) disjoint subsets of rational numbers.

Let $\mathbf{p}_\mathbb{Q}$ denotes the least cardinality of a family $D \subseteq D(\mathbb{Q})$ such that $\bigcap F \in D(\mathbb{Q})$ for each finite family $F \subseteq D$ and there is no $d \in D(\mathbb{Q})$ such that $d \subseteq^* x$ for each $x \in D$. Similarly, let $\mathbf{t}_\mathbb{Q}$ denotes the least cardinal number κ such that there exists a $\subseteq^*$-decreasing function $f : \kappa \to D(\mathbb{Q})$ such that there is no $d \in D(\mathbb{Q})$ such that $d \subseteq^* f(\alpha)$ for each $\alpha < \kappa$.

Now we prove the main result of this paper. A part of the proof of main theorem (inequality $\mathbf{t} \leq \mathbf{t}_Q$) was embedded in one proof of A. Szymański (see Theorem 2) and was independently rediscovered by G. Labedzki.

Theorem 1. $\mathbf{p} = \mathbf{p}_{\mathbb{Q}}$ *and* $\mathbf{t} = \mathbf{t}_{\mathbb{Q}}$.

Proof. We prove first that $\mathbf{p} \leqslant \mathbf{p}_{\mathbb{Q}}$. Suppose that $\kappa < \mathbf{p}$ and that $\langle D_\alpha : \alpha < \kappa \rangle$ is a centered family of dense subsets of rational numbers. Let $\langle I_n : n \in \omega \rangle$ be the family of all intervals with rational end-points. Let us fix $n \in \omega$. Then $\langle D_\alpha \cap I_n : \alpha < \kappa \rangle$ is a centered family of sets, hence there exists an infinite set $Y_n \subseteq I_n$ such that $(\forall \alpha < \kappa)(Y_n \subseteq^* D_\alpha)$. Let $\varphi_\alpha \in \omega^\omega$ be a function such that $Y_n \setminus \{q_k : k < \varphi_\alpha(n)\} \subseteq D_\alpha$ for each $n \in \omega$. Since $\mathbf{p} \leq \mathbf{b}$ there exists a function $\varphi \in \omega^\omega$ such that $(\forall \alpha < \kappa)(\varphi_\alpha \leq^* \varphi)$. Let $Y = \bigcup \{Y_n \setminus \{q_k : k < \varphi(n)\} : n \in \omega\}$. Then Y is a dense subset of $\mathbb{Q}$ and it is easy to check that $Y \subseteq^* D_\alpha$ for each $\alpha < \kappa$.

Next we show that $\mathbf{p}_{\mathbb{Q}} \leqslant \mathbf{p}$. Suppose hence that $\kappa < \mathbf{p}_{\mathbb{Q}}$ and that $\{X_\alpha : \alpha < \kappa\}$ is a family of infinite subsets of ω. Let $F : \mathbb{Q} \to \omega$ be a function from Lemma 1. Then $\{F^{-1}(X_\alpha) : \alpha < \kappa\}$ is a family of dense subsets of $\mathbb{Q}$ of cardinality less than $\mathbf{p}_{\mathbb{Q}}$. Let $D \in D(\mathbb{Q})$ be such that $D \subseteq^* F^{-1}(X_\alpha)$ for each $\alpha < \kappa$. Then $F(D)$ is an infinite set and $F(D) \subseteq^* X_\alpha$ for each $\alpha < \kappa$. The proof of the equality $\mathbf{t} = \mathbf{t}_Q$ is similar to the presented one. ∎

We shall give now one application of Theorem 1.

Theorem 2. *(Szymański)* $\mathbf{t} \leq add(\mathbb{K})$

Proof. Suppose that $\kappa < \mathbf{t}$ and that $\langle D_\alpha : \alpha < \kappa \rangle$ is a sequence of dense open subsets of the real line $\mathbf{R}$. By transfinite induction on $\alpha < \kappa$ we build a $\subseteq^*$-descending sequence $\langle S_\alpha : \alpha < \kappa \rangle$ of dense subsets of $\mathbb{Q}$ such that $S_\alpha \subseteq D_\alpha$ for each $\alpha < \kappa$. Since $\kappa < \mathbf{t} = \mathbf{t}_Q$, there exists a dense set $S \subseteq \mathbb{Q}$ such that $S \subseteq^* D_a$ for each $\alpha < \kappa$. Let $S = \{s_n\}_{n \in \omega}$. For each $\alpha < \kappa$ we find a function $\varphi_\alpha \in \omega^\omega$ such that the relation

$$\left(s_n - \frac{1}{\varphi_\alpha(n)}, s_n + \frac{1}{\varphi_\alpha(n)}\right) \subseteq D_\alpha$$

holds for every $n \in \omega$. Let us recall that the inequality $\mathbf{t} \leq \mathbf{b}$ holds (see [**1**]). Let $\varphi \in \omega^\omega$ be a function such that $(\forall \alpha < \kappa)(\varphi_\alpha \leq^* \varphi)$. Then it is easy to show that set

$$D = \bigcap_{k \in \omega} \bigcup_{n > k} \left(s_n - \frac{1}{\varphi_\alpha(n)}, s_n + \frac{1}{\varphi_\alpha(n)}\right)$$

is dense and is contained in the intersection of the sequence $\langle D_\alpha : \alpha < \kappa \rangle$. ∎

3. Dense embedding

Suppose that $S \subseteq [\omega]^\omega$. We say that the function $\varphi : \omega \to \mathbb{Q}$ is a *dense embedding of the family* S if φ is an injection and $\varphi(A) \in D(\mathbb{Q})$ for each $A \in S$. Let *deq* denotes the least cardinality of a family $S \subseteq [\omega]^\omega$ for which there is no dense embedding. It is easy to see that each countable family of infinite subsets of ω has a dense embedding and it is easy to see that there is no dense embedding of the whole family $[\omega]^\omega$. Therefore $\omega < deq \leq 2^\omega$.

Let us recall that a family $R \subseteq P(\omega)$ is a *reaping family* if for each set $A \subseteq \omega$ there exists $X \in R$ such that $X \subseteq A$ or $X \subseteq \omega \setminus A$. The least cardinality of a reaping family of subsets of ω is denoted by $\mathbf{r}$ (see [**1**] or [**4**]).

Lemma 2. *Suppose that $f, g \in \omega^\omega$ are strictly increasing, $N \in \omega$ are such that $(\forall n \geq N)\,(f(n) < g(n))$ and $g(0) > 0$. Then $(\forall n \geq N)\,(g^n(0) \leq f(g^n(0)) < g^{n+1}(n))$.*

Proof. Notice that $k \leq g^k(0), f(k)$ for each $k \in \omega$. Let $n \geq N$. Since f is strictly increasing, we have $g^n(0) \leq f(g^n(0))$. Moreover $g^n(0) \geq n$, so we have $f(g^n(0)) < g(g^n(n)) = g^{n+1}(n)$. ■

Theorem 3. $\max\{cov(\mathbb{K}), cov(\mathbb{L}), \mathbf{b}\} \leq deq \leq \mathbf{r}$

Proof. Suppose first that R is a reaping family of subsets of ω. Suppose that $\varphi : \omega \to \mathbb{Q}$ is a dense embedding of the family R. Let $A = \varphi^{-1}([0, \infty) \cap \mathbb{Q})$. Let $X \in R$ be such a set that $X \subseteq A$ or $X \subseteq \omega \setminus A$. If $X \subseteq A$ then $\varphi(X) \subseteq [0, \infty) \cap \mathbb{Q}$ and if $X \subseteq \omega \setminus A$ then $\varphi(X) \subseteq (-\infty, 0) \cap \mathbb{Q}$, so in both cases $\varphi(X)$ is not a dense subset of rational numbers. This shows that $deq \leq \mathbf{r}$.

Suppose now S is a family of infinite subsets of ω and that $|S| < cov(\mathbb{K})$. Let us treat $\mathbb{Q}$ as a discrete metric space and let us consider the product polish metric space $\mathbb{Q}^\omega$. Let $T = \{x \in \mathbb{Q}^\omega : (\forall n, m \in \omega)\,(n < m \to x(n) \neq x(m))\}$. Then T is a closed subset of the space $\mathbb{Q}^\omega$, hence is a polish space, too. Let us fix an enumeration $\{I_n\}_{n \in \omega}$ of all subintervals of $\mathbb{Q}$ with rational end-points. For each $A \in S$ and $n \in \omega$ we put $D_{A,n} = \{x \in T : (\exists k \in A)\,(x(k) \in I_n)\}$. Then $D_{A,n}$ are dense and open subsets of the space T. Hence the intersection $\bigcap\{D_{A,n} : A \in S \wedge n \in \omega\}$ is non-empty. It is easy to check that each element from this intersection is a required dense embedding of the family S. Hence we proved that $cov(\mathbb{K}) \leq deq$.

Let us assume now that S is a family of infinite subsets of ω and that $|S| < cov(\mathbb{L})$. Let $M_n = \{\frac{i}{p_n} : 0 < i < p_n\}$ where p_n is the n-th prime number. Let us consider a measure μ_n on the family of all subsets of the set M_n defined by the formula $\mu_n(A) = \frac{|A|}{p_n}$ and let us consider the product measure space $(M, \mu) = \prod_n (M_n, \mu_n)$. This measure space is isomorphic with the standard Lebesgue measure on the interval $[0, 1]$. Let us fix an enumeration $\{I_n\}_{n \in \omega}$ of all subintervals of $\mathbb{Q}$ with rational end-points. For each $A \in S$ and $n \in \omega$ we put $D_{A,n} = \{x \in T : (\exists k \in A)\,(x(k) \in I_n)\}$. Notice that

$$\mu(M \setminus D_{A,n}) = \prod_{k \in A} \mu_k(\{x \in M_k : x \notin I_n\} \cong (1 - |I_n|)^\infty = 0,$$

hence $\mu(D_{A,n}) = 1$. Therefore the intersection $\bigcap\{D_{A,n} : A \in S \wedge n \in \omega\}$ is non-empty. It is easy to check that each element from this intersection is a required dense embedding of the family S. Hence we proved that $cov(\mathbb{L}) \leq deq$.

Suppose now that $S \subseteq [\omega]^\omega$ and $|S < \mathbf{b}$. For each $A \in S$ let $f_A : \omega \to A$ be a strictly increasing surjection. Let $g \in \omega^\omega$ be are strictly increasing function such that $(\forall A \in S)(f_A \leq^* g)$ and $g(0) > 0$. Let $K_n = [g^n(0), g^{n+1}(n))$. Then for each $A \in S$ we have $(\exists N)(\forall n \geq N)(A \cap K_n \neq \emptyset)$. Let us fix an enumeration $\{I_n\}_{n\in\omega}$ of all subintervals of $\mathbb{Q}$ with rational end-points. Let $\{J_n\}_{n\in\omega}$ be a family of pairwise disjoint infinite sets such that $J_n \subset I_n$ for each $n \in \omega$. Finally, let $\varphi : \omega \to \mathbb{Q}$ be such that $\varphi(K_n) \subset J_n$. Then φ is a dense embedding of the family S. This shows that $\mathbf{b} \leq deq$. This finishes the proof of theorem. ■

We say that a set $X \in [\omega]^\omega$ *splits* a set $A \in [\omega]^\omega$ if $|A \cap X| = |A \setminus X| = \omega$. Let us recall the next two cardinal numbers used to describe combinatorial properties of the family of infinite subsets of $\omega : \mathbf{s} = \min\{|D| : (\forall A \in [\omega]^\omega)(\exists X \in D)(X \text{ splits } A)\}$, $\aleph_0 - \mathbf{s} = \min\{|D| : (\forall F \in ([\omega]^\omega)^\omega)(\exists X \in D)(\forall n)(X \text{ splits } F(n))\}$. Notice that $\mathbf{b} \leq \mathbf{s} \leq \aleph_0 - \mathbf{s}$ (see **[2]**). We define a splitting number for dense subsets of $\mathbb{Q}$ as follows: $\mathbf{s}_{\mathbb{Q}}$ is the least cardinality of a family D subsets of $\mathbb{Q}$ such that for each dense subset $A \subset \mathbb{Q}$ there exists $X \in D$ such that both sets $A \cap X$ and $A \setminus X$ are dense in $\mathbb{Q}$.

Theorem 4. $\mathbf{s}_{\mathbb{Q}} \leq \aleph_0 - \mathbf{s}$

Proof. Suppose that $D \subset [\omega]^\omega$ is a such family of sets that for each $F : \omega \to [\omega]^\omega$ there exists $X \in D$ such that $(\forall n)(X \text{ splits } F(n))$. Let $\varphi : \mathbb{Q} \to \omega$ be any bijection. Then the family $\{\varphi^{-1}(X) : X \in D\}$ splits all dense subsets of $\mathbb{Q}$ into dense sets. Indeed, let D be a dense subset $\mathbb{Q}$ and let $\{I_n\}_{n\in\omega}$ be an enumeration of all nonempty open subintervals with rational of $\mathbb{Q}$. Let us consider the function $F(n) = \varphi(D \cap I_n)$. There exists a set $X \in D$ such that $(\forall n \in \omega)(|F(n) \cap X| = |F(n) \setminus X| = \omega)$. Then $(\forall n \in \omega)(D \cap I_n \cap \varphi^{-1}(X) \neq \emptyset \wedge (D \cap I_n) \setminus \varphi^{-1}(X) \neq \emptyset)$. ■

Theorem 5. $\min\{deq, \mathbf{s}_{\mathbb{Q}}\} \leq \aleph_0 - \mathbf{s}$

Proof. Suppose that S is a family of subsets of ω such that $|S| < deq$ and $|S| < \mathbf{s}_{\mathbb{Q}}$. Let $\varphi : \omega \to \mathbb{Q}$ be a dense embedding of the family S. Then $|\{\varphi(X) : X \in S\}| < \mathbf{s}_{\mathbb{Q}}$ so the family S does not densely split all dense subsets of $\mathbb{Q}$, i.e. there exists a dense subset $A \subset \mathbb{Q}$ such that

$$(\forall X \in S)(A \cap \varphi(X) \notin D(\mathbb{Q}) \wedge A \setminus \varphi(X) \notin D(\mathbb{Q})).$$

Let $\{I_n\}_{\in\omega}$ be an enumeration of all subintervals of $\mathbb{Q}$. Then we have

$$(\forall X \in S)(\exists n \in \omega)(|(A \cap I_n) \cap \varphi(X)| < \omega \wedge |(A \cap I_n) \setminus \varphi(X)| < \omega),$$

so

$$(\forall X \in S)(\exists n \in \omega)(|A_n \cap X| < \omega \wedge |A_n \setminus X| < \omega),$$

where $A_n = \varphi^{-1}(A \cap I_n)$. Note that A_n is an infinite set for each $n \in \omega$. Therefore S is not an $\aleph_0$ splitting family for infinite subsets of ω. ■

References

[1] van Douwen E. K., *The integers and topology*, Handbook of Set Theoretical Topology (K. Kunen and J. Vaughan, eds), North-Holland, Amsterdam **1984**, pp. 111 – 167.

[2] Kamburelis A., Weglorz B., *Splittings*, Arch. Math. Logic **35** (1996), pp. 263 – 277.

[3] Sierpiński W., *Cardinal and Ordinal Numbers*, Państwowe Wydawnictwa Naukowe, Warszawa **1958**.

[4] Vaughan J., *Small Uncountable Cardinals and Topology*, Open Problems in Topology (j. van Mill and G. Reed, eds), North Holland, Amsterdam, **1990**, pp. 195 – 218.