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RECONSTRUCTING EQUIVALENCES

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Abstract: A graph is called an equivalence if each of its components of connectivity is a complete graph. We ask whether an equivalence is uniquely determined with its k-point subobjects. For each k we prove: 1/Every equivalence on less than k.ln(k/2)-point set is uniquely determined with k-point subobjects; 2/It is not true that every equivalence on at least $(k+1).2^{k-1}$-point set is uniquely determined with its k-point subobjects.

0. Introduction

We denote $\langle V,W\rangle$ the ordered pair where the first member is V and the second one is W. $P_2(X)$ denotes the set of all 2-point subsets of the set X. An ordered pair $G = \langle X,R\rangle$ where $R \subset P_2(X)$ is called a graph and we denote $|G| = \text{card } X$ the number of points of X. The complete graph on X is the graph $\langle X,P_2(X)\rangle$ and we denote K_n the standard complete graph on n-point set. For the graph $G = \langle X,R\rangle$ and the set $Y \subset X$ we define the induced graph $G/Y = \langle Y,R \cap P_2(Y)\rangle$. In usual sense we work with concepts in graph theory, namely the connectivity of graphs, components of connectivity, isomorphism of graphs. The number of components of G is denoted cp G; isomorphic graphs are denoted $G \simeq H$ and nonisomorphic graphs $G \not\simeq H$.

For every sequence of complete graphs $K_{n_1}, \dots, K_{n_s}$ it is the standard sum $K = K_{n_1} + \dots + K_{n_s}$ with components of connectivity $C_1, \dots, C_s$ satisfying $K/C_i \simeq K_{n_i}$; if $n_1 = \dots = n_s = n$ we write simply $K = s.K_n$.

Definition 0.1. A graph E is called an equivalence if E is isomorphic to a sum of complete graphs.

Definition 0.2. The frequency of the graph H in the graph G

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is the number $\mathrm{frq}(H,G) = \mathrm{card}\ \{Y;\ G/Y \simeq H\}$. For an integer k the notation $G_1 \overset{k}{\sim} G_2$ /$G_1 \overset{\le k}{\sim} G_2$, respecively/ means that for every graph H such that $|H| = k$ /$|H| \le k$, respectively/ the equality $\mathrm{frq}(H,G_1) = \mathrm{frq}(H,G_2)$ holds.

Remark 0.3. An induced graph of an equivalence is an equivalence. Thus, if E is an equivalence then $\mathrm{frq}(H,E) > 0$ if and only if H is an equivalence.

We have showed in [6] the following theorem.

Theorem 0.4. Let k be an integer, G_1, G_2 be graphs. Following three properties are equivalent /i/ $G_1 \overset{k}{\sim} G_2$, /ii/ $G_1 \overset{\le k}{\sim} G_2$, /iii/ for every connected graph $H, |H| \le k$, it is $\mathrm{frq}(H,G_1) = \mathrm{frq}(H,G_2)$

Now, for the case of equivalences we get:

Theorem 0.5. Let k be an integer, E_1, E_2 be equivalences. Following three properties are equivalent /i/ $E_1 \overset{k}{\sim} E_2$, /ii/ $E_1 \overset{\le k}{\sim} E_2$, /iii/ for every $j \le k$ $\mathrm{frq}(K_j,E_1) = \mathrm{frq}(K_j,E_2)$.

Proof. Use Remark 0.3., Theorem 0.4. and the fact, that only complete graphs are connected equivalences.

1. Frequencies in equivalences

Throughout this part of paper let A, B, C be equivalences, $A = s.K_u$, $B = K_v$ where $s > 0$, $u > 0$, $v > 0$, $Q = s.K_u + K_v$ and for every $i \le u+v$ $Q_i = (s-1).K_u + K_i$.

Definition 1.1. We define two numbers for any equivalence E
$\langle A,B\rangle{\downarrow}E = \mathrm{card}\ \{\langle Y,Z\rangle;\ E/Y \simeq A, E/Z \simeq B\} = \mathrm{frq}(A,E).\mathrm{frq}(B,E)$,
$\langle A,B\rangle{\downarrow\downarrow}E = \mathrm{card}\{\langle Y,Z\rangle;\ E/Y \simeq A, E/Z \simeq B, Y \cup Z = X\}$, where $E = \langle X,R\rangle$.

Lemma 1.2. Let E be an equivalence. Then $\mathrm{card}\{\langle Y,Z\rangle;\ C/Y \simeq A, C/Z \simeq B, C/(Y \cup Z) \simeq E\} = [\langle A,B\rangle{\downarrow\downarrow}E].\mathrm{frq}(E,C)$.

Proof. The number of the sets W such that $C/W \simeq E$ is $\mathrm{frq}(E,C)$. For each such a set W we have $\langle A,B\rangle{\downarrow\downarrow}E$ ordered pairs $\langle Y,Z\rangle$ satisfying $C/Y \simeq A$, $C/Z \simeq B$, $W = Y \cup Z$.

Remark 1.3. Equivalences A,B in Lemma 1.2. can be arbitrary.

Lemma 1.4. The following equality is true

$$\langle A,B\rangle{\downarrow}C = \sum_{i=1}^{u+v-1} [\langle A,B\rangle{\downarrow\downarrow} Q_i].\mathrm{frq}(Q_i,C) + [\langle A,B\rangle{\downarrow\downarrow}Q_{u+v}].\mathrm{frq}(Q_{u+v},C) + [\langle A,B\rangle{\downarrow\downarrow}Q].\mathrm{frq}(Q,C)\ .$$

Proof. We denote $M_0 = \{\langle Y,Z\rangle;\ C/Y \simeq A, C/Z \simeq B\}$ and further for every $i \le u+v$ $M_i = \{\langle Y,Z\rangle;\ C/Y \simeq A, C/Z \simeq B, C/(Y \cup Z) \simeq Q_i\}$. Finally, $M = \{\langle Y,Z\rangle;\ C/Y \simeq A, C/Z \simeq B, C/(Y \cup Z) \simeq Q\}$. We have the disjoint decomposition $M_0 = M_1 \cup \dots \cup M_{u+v-1} \cup M_{u+v} \cup M$ and we can write $\mathrm{card}\ M_0 = \sum_{i=1}^{u+v-1} \mathrm{card}\ M_i + \mathrm{card}\ M_{u+v} + \mathrm{card}\ M$. Using Lemma 1.2. we obtain the needed equality.

Lemma 1.5. Let $j+1 = s.u+v$, let E_1,E_2 be two equivalences such that $E_1 \overset{j}{\sim} E_2$ and $frq(Q,E_1) = frq(Q,E_2)$. Then $frq(Q_{u+v},E_1) = frq(Q_{u+v},E_2)$.

Proof. For $i \le u+v-1$ it is $|Q_i| = (s-1).u+i \le s.u+v-1 = j$ and by Theorem 0.5. $frq(Q_i,E_1) = frq(Q_i,E_2)$. Analogiously, since $v \le j$ and $s.u \le j$ we have $frq(K_v,E_1) = frq(K_v,E_2)$ and $frq(s.K_u,E_1) = frq(s.K_u,E_2)$, i.e. $\langle s.K_u,K_v\rangle \downarrow E_1 = \langle s.K_u,K_v\rangle \downarrow E_2$. Now, we calculate using Lemma 1.4. $0 = \langle s.K_u,K_v\rangle \downarrow E_1 - \langle s.K_u,K_v\rangle \downarrow E_2 = \sum_{i=1}^{u+v-1} \langle s.K_u,K_v\rangle \downarrow\downarrow Q_i.[frq(Q_i,E_1) - frq(Q_i,E_2)] + \langle s.K_u,K_v\rangle \downarrow\downarrow Q_{u+v}.[frq(Q_{u+v},E_1) - - frq(Q_{u+v},E_2)] + \langle s.K_u,K_v\rangle \downarrow\downarrow Q.[frq(Q,E_1) - frq(Q,E_2)] = \langle s.K_u,K_v\rangle \downarrow\downarrow Q_{u+v}.[frq(Q_{u+v},E_1) - frq(Q_{u+v},E_2)]$. Since $\langle s.K_u,K_v\rangle \downarrow\downarrow Q_{u+v} > 0$ we get finally $frq(K_{u+v},E_1) = frq(K_{u+v},E_2)$.

Definition 1.6. An equivalence E is called pseudoregular if there exist numbers $s \ge 0$, $u > 0$, $v > 0$ such that $E \simeq s.K_u + K_v$.

Now, we are able to prove the main theorem.

Theorem 1.7. Let k be an integer, E_1,E_2 be equivalences. Following four properties are equivalent /i/ $E_1 \overset{k}{\sim} E_2$, /ii/ $E_1 \overset{\le k}{\sim} E_2$, /iii/ for every $j \le k$ $frq(K_j,E_1) = frq(K_j,E_2)$, /iv/ for every $j \le k$ there is a pseudoregular equivalence S_j such that $|S_j| = j$ and $frq(S_j,E_1) = frq(S_j,E_2)$.

Proof. To prove the theorem it suffices to show that the implication /iv/$\Rightarrow$ /iii/ is true. We use an indirect argument. If the implication is false there exist $i \le k$ such that $frq(K_i,E_1) \ne \ne frq(K_i,E_2)$. Let $i^* = \min\{i;\ frq(K_i,E_1) \ne frq(K_i,E_2)\}$. Obviously $1 < i^* \le k$ and for $j = i^* - 1$ we have by Theorem 0.5. $E_1 \overset{j}{\sim} E_2$. We know that $frq(S_{j+1},E_1) = frq(S_{j+1},E_2)$. Let $c = \min\{cp\ S;\ S$ is pseudoregular, $|S| = j+1$, $frq(S,E_1) = frq(S,E_2)\}$. Then $1 < c \le cp\ S_{i^*}$. Take $Q = s.K_u + K_v$ such that $s > 0$, $u > 0$, $v > 0$, $|Q| = j+1$, $cp\ Q = = c = s+1$. By Lemma 1.5. $frq((s-1).K_u + K_{u+v},E_1) = frq((s-1).K_u + + K_{u+v},E_2)$ contradicting the minimality of c because $cp\,[(s-1).K_u + + K_{u+v}] = s < c$.

Theorem 1.8. Let $k > 0$, E_1,E_2 be equivalences, $E_1 \overset{k}{\sim} E_2$. If there exists a pseudoregular equivalence S such that $|S| \le k$ and $frq(S,E_1) = frq(S,E_2) = 0$ then $E_1 \simeq E_2$.

Proof. Let $n = |E_1| = |E_2|$, let $S = s.K_u + K_v$, $|S| \le k$, $frq(S,E_1) = frq(S,E_2)$. For every integer w define $S_w = s.K_u + K_w$. For $w \le v$ we have $frq(S_w,E_1) = frq(S_w,E_2) = 0$ and by Theorem 1.7. /property /iv// $E_1 \overset{n}{\sim} E_2$. It is $1 = frq(E_1,E_1) = frq(E_1,E_2)$ and clearly $E_1 \simeq E_2$.

2. Bounds of reconstructibility and nonreconstructibility

We are interested in the problem: for given k find n satisfying the implication $(|E_1| = |E_2| = n$ et $E_1 \overset{k}{\sim} E_2) \Rightarrow (E_1 \simeq E_2)$ where E_1, E_2 are arbitrary equivalences.

We denote $cp_i E$ the number of components of the equivalence E having at least i elements. Let us indicate two elementary facts: /fact 1/ $|E| = \sum_{i\geq 1} cp_i E$, /fact 2/ if $frq(s.K_i, E) \geq 1$ then $cp_i E \geq s$.

<u>Theorem 2.1.</u> Let $k > 2$, E_1, E_2 be equivalences, $|E_1| = |E_2| \leq \leq k.\ln(k/2)$ where ln denotes the logarithmus naturalis. If $E_1 \overset{k}{\sim} E_2$ then $E_1 \simeq E_2$.

Proof. Suppose $E_1 \not\simeq E_2$ and define for every $i \leq k$ the integral part of k/i denoted $t_i = [k/i]$. Now, for every $i \leq k$ we have frq $(t_i.K_i, E_1) \geq 1$ by Theorem 1.8. and moreover $cp_i E_1 \geq t_i$ by /fact 2/. We calculate $n = |E_1| = \sum_{i\geq 1} cp_i E_1 \geq \sum_{i=1}^{k} t_i \geq \sum_{i=1}^{k} (k/i - 1) = (k.\sum_{i=1}^{k-1} 1/i) + + 1 > k.\ln(k/2) + 1$. We get a contradiction with the assumption that $n \leq k.\ln(k/2)$.

<u>Construction 2.2.</u> For every $k \geq 1$ we construct two equivalences E_1, E_2 such that $E_1 \overset{k}{\sim} E_2$, $E_1 \not\simeq E_2$, $|E_1| = |E_2| = (k+1).2^{k-1}$.

Proof. For $i=1,\dots,k+1$ we define the numbers a_i, b_i

$$a_i = \begin{cases} \binom{n+1}{i} & \text{if i is even} \\ 0 & \text{if i is odd} \end{cases} \qquad b_i = \begin{cases} 0 & \text{if i is even} \\ \binom{n+1}{i} & \text{if i is odd .} \end{cases}$$

The numbers a_i, b_i satisfy $a_i - b_i = (-1)^i \binom{n+1}{i}$, $a_i + b_i = \binom{n+1}{i}$.

We define $E_1 = \sum_{i=1}^{k+1} a_i.K_i$, $E_2 = \sum_{i=1}^{k+1} b_i.K_i$. It is obvious that $E_1 \not\simeq E_2$ because E_2 has 1-point components but E_1 has not. For every j, $1 \leq j \leq k$ we calculate $frq(K_j, E_1) - frq(K_j, E_2) = \sum_{i=j}^{k+1} a_i.\binom{i}{j} - \sum_{i=j}^{k+1} b_i.\binom{i}{j} = \sum_{i=j}^{k+1} (a_i - b_i).\binom{i}{j} = \sum_{i=j}^{k+1} (-1)^i.\binom{k+1}{i}.\binom{i}{j} = 0$ and we get $frq(K_j, E_1) = = frq(K_j, E_2)$. It is $E_1 \overset{k}{\sim} E_2$ by Theorem 1.7.

Finally, we calculate $|E_1| + |E_2| = \sum_{i=1}^{k+1} a_i.i + \sum_{i=1}^{k+1} b_i.i = = \sum_{i=1}^{k+1} (a_i + b_i).i = \sum_{i=1}^{k+1} \binom{k+1}{i}.i = (k+1).2^k$, which yields $|E_1| = |E_2| = = (k+1).2^{k-1}$.

<u>Remark 2.3.</u> In [6] we have defined reconstructibility indicating function $u_{\mathcal{E}}$ of the class of graphs $\mathcal{E}$. If we denote $\mathcal{E}$ the class of all equivalences we can write the result of this paper in the form: for every $k > 2$ $k.\ln(k/2) \leq u_{\mathcal{E}}(k) < (k+1).2^{k-1}$.

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