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Embedding Topological Semigroups into the Hyperspaces over Topological Groups

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We study algebraic and topological properties of subsemigroups of the hyperspaces $\exp(G)$ of non-empty compact subsets of a topological group G endowed with the Vietoris topology and the natural semigroup operation. On this base we prove that a compact Clifford topological semigroup S is topologically isomorphic to a subsemigroup of $\exp(G)$ for a suitable topological group G if and only if S is a topological inverse semigroup with zero-dimensional idempotent semilattice.

1. Introduction

According to [Ber] (and [Trn]) each (commutative) semigroup S embeds into the global semigroup $\Gamma(G)$ over a suitable (Abelian) group G. The global semigroup $\Gamma(G)$ over G is the set of all non-empty subsets of G endowed with the semigroup operation $(A, B) \mapsto AB = \{ab : a \in A,\ b \in B\}$. If G is a topological group, then the global semigroup $\Gamma(G)$ contains a subsemigroup $\exp(G)$ consisting of all non-empty compact subsets of G and carrying a natural topology turning it into a topological semigroup. This is the Vietoris topology generated by the sub-base consisting of the sets

$$U^+ = \{K \in \exp(S) : K \subset U\} \text{ and } U^- = \{K \in \exp(S) : K \cap U \neq \emptyset\}$$

where U runs over open subsets of S. Endowed with the Vietoris topology the semigroup $\exp(G)$ will be referred to as the *hypersemigroup* over G (because its

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underlying topological space is the hyperspace $\exp(G)$ of G, see [TZ]). Since each topological group G is Tychonov, so is the hypersemigroup $\exp(G)$. The group G can be identified with the subgroup $\{K \in \exp(G) : |K| = 1\}$ of $\exp(G)$ consisting of singletons.

The main object of our study in this paper is the class $\mathscr{H}$ of topological semigroups S that embed into the hypersmigroups $\exp(G)$ over topological groups G. We shall say that a topological semigroup S_1 embeds into another topological semigroup S_2 if there is a semigroup homomorphism $h : S_1 \to S_2$ that is a topological embedding. In is clear that the class $\mathscr{H}$ contains all topological groups. On the other hand, the compact topological semigroup $([0,1], \min)$ does not belong to $\mathscr{H}$, see [BL]. In this paper we establish some inheritance properties of the class $\mathscr{H}$ and on this base detect compact Clifford semigroups belonging to $\mathscr{H}$: those are precisely compact Clifford inverse semigroups with zero-dimensional idempotent semilattice.

Let us recall that a semigroup S is

- *Clifford* is each element $x \in S$ lies in a subgroup of S;
- *inverse* if each element $x \in S$ is *uniquely invertible* in the sense that there is a unique element $x^{-1} \in S$ called the *inverse* of x such that $xx^{\ 1}x = x$ and $x^{\ 1}xx^{\ 1} = x^{\ 1}$;
- *algebraically regular* if each element $x \in S$ is *regular* in the sense that $xyx = x$ for some $y \in S$;
- a *semillattice* if $xx = x$ and $xy = yx$ for all $x, y \in S$.

It is known [CP, 1.17], [Pet, II.1.2] that a semigroup S is inverse if and only if S is algebraically regular and the set $E = \{x \in S : xx = x\}$ of idempotents is a commutative subsemigroup of S. The subsemigroup E will be called the *idempotent semilattice* of S. An inverse semigroup S is Clifford if and only if $xx^{\ 1} = x^{-1}x$ for all $x \in S$. In this case $S = \bigcup_{e \in E} H_e$ where $H_e = \{x \in S : xx^{\ 1} = = e = x^{-1}x\}$ are the maximal subgroups of S corresponding to the idempotents e of S.

The above classes of semigroups relate as follows:

$$\begin{array}{ccccccc}
\text{group} & & & & \text{inverse} & & \\
& \searrow & & \nearrow & & \searrow & \\
& & \text{Clifford inverse} & & & & \text{algebraically regular} \\
& \nearrow & & \searrow & & \nearrow & \\
\text{semilattice} & & & & \text{Clifford} & &
\end{array}$$

These classes form varieties of semigroups, which means that they are closed under taking subdirect products and homomorphic images. As we shall see later, the class $\mathscr{H}$ is not closed under homomorphism images and thus does not form a variety but is invariant with respect to many operations over topological semigroups.

By a *topological semigroup* we understand a topological space S endowed with a continuous semigroup operation. A topological semigroup S is called a *topological inverse semigroup if* S is an inverse semigroup and the inversion map $(\cdot)^{-1}: S \to S$, $(\cdot)^{-1}: x \mapsto x^{-1}$ is continuous.

Now we define three operations over topological semigroups that do not lead out the class $\mathscr{H}$.

We say that a topological semigroup S is a *subdirect product* of a family $\{S_\alpha : \alpha \in A\}$ of a topological semigroups if S embeds into the Tychonov product $\prod_{\alpha \in A} S_\alpha$ endowed the coordinatewise semigroup operation.

Another operation is the *semidirect product* $S \rtimes^\sigma G$ of a topological semigroup S and a topological group G acting on S by authomorphism. More precisely, let $Aut(S)$ denote the group of topological auto-isomorphisms of the semigroup S and $\sigma : G \to Aut(S)$ be a group homomorphism defined on a topological group G and such that the induced map

$$\tilde{\sigma} : G \times S \to S, \ \tilde{\sigma} : (g, s) \mapsto \sigma(g)(s)$$

is continuous. By $S \rtimes^\sigma G$ we denote the topological semigroup whose underlying topological space is the Tychonov product $S \times G$ and the semigroup operation is given by the formula $(s, g) * (s', g') = (sg(s'), gg')$. The semidirect product $S \rtimes^{id} Aut(S)$ of a semigroup S with its automorphism group is called *the holomorph* of S and is denoted by $\mathrm{Hol}(S)$.

One can easily check that for an algebraically regular (inverse) topological semigroup S the semidirect product $S \rtimes^\sigma G$ with any topological group G acting on S is an algrabraically regular (inverse) topological semigroup. The situation is different for Clifford topological semigroups: the semidirect product $S \rtimes^\sigma G$ is Clifford inverse if and only if S is Clifford inverse and the group G acts trivially on the idempotents of S, see Proposition 3.

The third operation that does not lead out the class $\mathscr{H}$ is attaching zero to a compact semigroup from $\mathscr{H}$. Given a topological semigroup S let $S^0 = S \cup \{0\}$ denote the extension of S by an isolated point $0 \notin S$ such that $s0 = 0s = 0$ for all $s \in S^0$.

Theorem 1. *The class $\mathscr{H}$ is closed under the following three operations:*
(1) *subdirect products;*
(2) *semidirect products with Abelian topological groups;*
(3) *attaching zero to compact semigroups from $\mathscr{H}$.*

Proof. 1. The first item follows from the fact that for any family $\{H_\alpha\}_{\alpha \in A}$ of topological groups the map

$$E : \prod_{\alpha \in A} \exp(H_\alpha) \to \exp\left(\prod_{\alpha \in A} H_\alpha\right), \qquad E : (K_\alpha)_{\alpha \in A} \mapsto \prod_{\alpha \in A} K_\alpha$$

is an embedding of topological semigroups.

2. The second item is less trivial an will be proved in Section 2.

3. If $S \in \mathscr{H}$ is a compact semigroup, then there is an embedding $f : S \to \exp(H)$

of S into the hypersemigroup $\exp(H)$ of some compact topolgical group H, see Proposition 1 below. Take any compact topological group G containing H so that $H \neq G$ and define the map $f^0 : S^0 \to \exp(G)$ letting $f^0 \mid S = f$ and $f^0(0) = G$. It can be shown that f^0 is a topological embedding and thus $S^0 \in \mathcal{H}$. $\square$

Problem 1. *Is the class $\mathcal{H}$ closed under taking semidirect products with arbitrary (not necessarily Abelian) topological groups?*

In light of Theorem 1 it is natural to consider the smallest class $\mathcal{H}_0$ of topological semigroups, closed under subdirect products, semidirect products with Abelian topological groups and attaching zero to compact semigroups from $\mathcal{H}$. Since the class of topological inverse semigroups is closed under those three operations, we conclude that $\mathcal{H}_0$ is a subclass of the class of topological inverse semigroups. Consequently, $\mathcal{H}_0$ is strictly smaller that the class $\mathcal{H}$ (because for a topological group G the semigroup $\exp(G)$ is inverse if and only if $|G| \leq 2$.

Nonetheless we can ask the following

Question 1. *Does each (compact) topological inverse semigroup $S \in \mathcal{H}$ belong to the class $\mathcal{H}_0$?*

In this respect let us note the following property of compact semigroups from the class $\mathcal{H}$.

Proposition 1. *A compact topological semigroup S belongs to the class $\mathcal{H}$ if and only if S embeds into the hypersemigroup* $\exp(G)$ *over a compact topological group G.*

Proof. Given a compact topological semigroup $S \in \mathcal{H}$ find an embedding $h : S \to \exp(G)$ of S into the hypersemigroup $\exp(G)$ over a topological group G. It follows from [TZ, 2.1.2] that the union $H = \bigcup_{s \in S} h(s) \subset G$ is compact. Moreover, H is a subsemigroup of G. Indeed, given arbitrary points $y, y' \in H$ find points $x, x' \in S$ with $y \in h(x)$ and $y' \in h(x')$. Then $yy' \in h(x)h(x') = h(xx') \subset H$. Being a compact cancellative semigroup, H is a topological group by [CHK1, Th.1.10]. Since $h(S) \subset \exp(H) \subset \exp(G)$, we see that S embeds into the hypersemigroup $\exp(H)$ over the compact topological group H. $\square$

We shall affirmatively answer the "compact" part of Question 1 under an additional assumption that $S \in \mathcal{H}$ is Clifford. For this we first establish some specific algebraic and topological properties of algebraically regular semigroups $S \in \mathcal{H}$.

Let us call two elements x, y of an inverse semigroup S *conjugated* if $x = zyz^{-1}$ and $y = z^{-1}xz$ for some element $z \in S$. For an element $e \in E$ of a semilattice E let $\uparrow e = \{f \in E : ef = e\}$ denote the principal filter of e. We say that two elements $e, f \in E$ are *incomparable* if their product ef differs from e and f (this is equivalent to $e \notin \uparrow f$ and $f \notin \uparrow e$).

A topological space X is called

- *totally disconnected* if for any distinct points $x, y \in X$ there is a closed-and-open subset $U \subset X$ containing x but not y;
- *zero-dimensional* if the family of closed-and-open sets forms a base of the topology of X.

It is known that a compact Hausdorff space is zero-dimensional if and only if it is totally disconnected.

Theorem 2. *If a topological semigroup $S \in \mathscr{H}$ is algebraically regular, then*
(1) *S is a topological inverse semigroup;*
(2) *the idempotent semilattice E of S has totally disconnected principal filters $\uparrow e, e \in E$;*
(3) *an element $x \in S$ is an idempotent if and only if x^2x^{-1} is an idempotent;*
(4) *any distinct conjugated idempotents of S are incomparable.*
This theorem will be proved in Section 3.

Remark 1. Theorem 2 allows us to construct many examples of algebraically regular topological semigroups non-embeddable into the hypersemigroups over topological groups. The first two items of this proposition imply the result of [BL] that non-trivial rectangular semigroups and connected topological semilattices do not belong to the class $\mathscr{H}$. The last two items imply that the class $\mathscr{H}$ does not contain neither Brandt nor bicyclic semigroup. A *bicyclic semigroup* is a semigroup generated by two elements p, q connected by the relation $qp = 1$.

By a *Brandt semigroup* we understand a semigroup of the form

$$B(H, \kappa) = (\kappa \times H \times \kappa) \cup \{0\}$$

where H is a group, κ is a non-empty set, and the product $(\alpha, h, \beta) * (\alpha', h', \beta')$ of two non-zero elements of $B(H, \kappa)$ is equal to (α, hh', β') if $\beta = \alpha'$ and 0 otherwise. Brandt and bicyclic semigroups play an important role in the structure theory of inverse semigroups, see [Pet].

The following theorem answers affirmatively the "compact" part of Question 1.

Theorem 3. *For a compact topological Clifford semigroup S the following conditions are equivalent:*
(1) *S belongs to the class $\mathscr{H}$;*
(2) *S belongs to the class $\mathscr{H}_0$;*
(3) *S is a topological inverse semigroup with zero-dimensional idempotent semilattice E;*
(4) *S embeds into the product $\prod_{e \in E} H_e^\circ$;*
(5) *S embeds into the hypersemigroup* $\exp(G)$ *of the compact topological group $G = \prod_{e \in E} \tilde{H}_e$, where for each idempotent $e \in E$ $\tilde{H}_e$ is a non-trivial compact topological group containing the maximal group H_e.*

Proof. It suffices to prove the implications: $(4) \Rightarrow (5) \Rightarrow (2) \Rightarrow (1) \Rightarrow (3) \Rightarrow (4)$ among which $(5) \Rightarrow (2) \Rightarrow (1)$ are trivial.

$(4) \Rightarrow (5)$. Assume that S embeds into the product $\prod_{e\in E} H_e^0$. For each idempotent $e \in E$ fix a non-trivial compact topological group $\tilde{H}_e$ containing H_e and define an embedding $f_e : H_e^0 \to \exp(\tilde{H}_e)$ letting $f_e(h) = \{h\}$ if $h \in H_e$ and $f_e(0) = \tilde{H}_e$.

The product of embeddings f_e, $e \in E$, yields embeddings

$$S \to \prod_{e\in E} H_e^0 \to \prod_{e\in E} \exp(\tilde{H}_e) \to \exp\left(\prod_{e\in E} \tilde{H}_e\right)$$

the latter homomorphism defined by

$$\prod_{e\in E} \exp(\tilde{H}_e) \ni (K_e)_{e\in E} \mapsto \prod_{e\in E} K_e \in \exp\left(\prod_{e\in E} \tilde{H}_e\right)$$

$(1) \Rightarrow (3)$ Assume that $S \in \mathscr{H}$. Then S is a compact topological inverse semigroup according to Theorem 2(1). The semigroup E of idempotents of S is compact and thus contains the smallest idempotent $e \in E$ (in the sense that $ee' = e$ for all $e' \in E$). By Theorem 2, the principal filter $\uparrow e = E$ is totally disconnected and being compact, is zero-dimensional.

$(3) \Rightarrow (4)$ Assume that S is a compact topological inverse Clifford semigroup with zero-dimensional idempotent semilattice E. Let $\pi : S \to E$, $\pi : x \mapsto xx^{-1} = x^{-1}x$ be the retraction of S onto E. The set E carries a natural partial order $\leq : e \leq e'$ iff $ee' = e$. Let $E_0 = \{e \in E : \uparrow e \text{ is open}\}$ stands for the set of locally minimal elements of E.

For every $e \in E \setminus E_0$ let $h_e : S \to H_e^0$ be the trivial homomorphism mapping S into the zero of H_e^0.

Next, for every $e \in E_0$ consider the homomorphism $h_e : S \to H_e^0$ defined by

$$h_e(s) = \begin{cases} es, & \text{if } s \in \pi^{-1}(\uparrow e); \\ 0, & \text{otherwise} \end{cases}$$

Taking the diagonal product of the homomorphisms h_e, $e \in E$, we obtain a homomorphism

$$h = (h_e)_{e\in E} : S \to \prod_{e\in E} H_e^0, \qquad h : s \mapsto (h_e(s))_{e\in E}.$$

We claim that h is injective and thus an embedding of the compact semigroup S into $\prod_{e\in E} H_e^0$.

Let $x, y \in S$ be two distinct points. If $\pi(x) \neq \pi(y)$ then either $\pi(x) \notin \uparrow\pi(y)$ or $\pi(y) \notin \uparrow\pi(x)$. We lose no generality assuming the first case. Consider the set $U = \{u \in E : \pi(x) \notin \uparrow u\}$ and note that it is open and $U = \uparrow U$ where $\uparrow U = \{v \in E : \exists u \in E \text{ with } u \leq v\}$. Also $\pi(y) \in U$. By Proposition 1 of [Hr] there is a continuous semilattice homomorphism $h : E \to \{0,1\}$ such that $\pi(y) \in h^{-1}(1) \subset$

$\subset \uparrow U$. The preimage $h^{-1}(1)$, being a compact subsemilattice of E, has the smallest element e, that belongs to E_0 because $h^{-1}(1) = \uparrow e$.

Now the definition of the homomorphism h_e and the non-inclusion $\pi(x) \notin \uparrow e$ imply that $h_e(x) = 0$ while $h_e(y) \in H_e$. Hence $h_e(x) \neq h_e(y)$ and $h(x) \neq h(y)$.

Finally consider the case $\pi(x) = \pi(y)$. Observe that the set $U = \{e \in E : xy \neq ye\}$ contains the idempotent $\pi(x) = \pi(y)$ and coincides with $\uparrow U$. Again applying Proposition 1 of [Hr] we can find a continuous semilattice homomorphism $h : E \to \{0,1\}$ such that $\pi(x) = \pi(y) \in h^{-1}(1) \subset \uparrow U$. The preimage $h^{-1}(1)$, being a compact subsemilattice of E, has the smallest element e. Since $h^{-1}(1) = \uparrow e$ is open in E, $e \in E_0$. It follows from $e \in U$ that $h_e(x) = ex \neq ey = h_e(y)$ and hence $h(x) \neq h(y)$. □

Theorem 3 will be applied to characterize Clifford compact topological semigroups embeddable into the hypersemigroups of topological groups G belonging to certain varieties of compact topological groups. A class $\mathcal{G}$ of topological groups is called a *variety* if it is closed under taking arbitrary Tychonov products, taking closed subgroups, and quotient groups by closed normal subgroups.

Theorem 4. *Let $\mathcal{G}$ be a non-trivial variety of compact topological groups. A Clifford compact topological semigroup S embeds into the hypersemigroup* $\exp(G)$ *of a topological group $G \in \mathcal{G}$ if and only if S is a topological inverse semigroup whose idempotent semilattice E is zero-dimensional and all maximal groups H_e, $e \in E$, belong to the class $\mathcal{G}$.*

This theorem will be proved in Section 4 after establishing the nature of group elements in the hypersemigroups.

The classes $\mathcal{H}$ and $\mathcal{H}_0$ are closed under subdirect products but are very far from being closed under homomorphic images. We shall show that the class of continuous homomorphic images of compact Clifford semigroups $S \in \mathcal{H}_0$ coincides with the class of all compact Clifford inverse semigroups with Lawson idempotent semilattices. We recall that a topological semilattice E is called *Lawson* if open subsemilattices form a base of the topology of E. By the fundamental Lawson Theorem [CHK2, Th. 2.13] a compact topological semilattice is Lawson if and only if the continuous homomorphisms to the min-interval $[0, 1]$ separate points of S. It is known [CHK2, Th. 2.6] that each zero-dimensional compact topological semilattice is Lawson.

Proposition 2. A *topological semigroup S is a continuous homomorphic image of a compact Clifford semigroup $S_0 \in \mathcal{H}_0$ if and only if S is a compact Clifford topological inverse semigroup with Lawson idempotent semilattice.*

Proof. To prove the "only if" part, assume that a topological semigroup S is the image of a compact Clifford semigroup $S_0 \in \mathcal{H}_0$ under a continuous homomorphism $h : S_0 \to S$. By Theorem 3(3), S_0 is a topological inverse Clifford semigroup with zero-dimensional idempotent semilattice E_0. Then S is an inverse Clifford semigroup, being the homomorphism image of S_0, see [Pet, L.II.1.10]. Moreover, being compact topological semigroup, S is a topological inverse semigroup, see [KW], [Kr] or [BG]. It follows that the semigroup E of idempotents of S is the homomorphic image of the semilattice E_0. Being zero-dimensional and compact, the semilattice E_0 is Lawson [CHK2, Th.2.6]. Then E is Lawson as the compact homomorphic image of a Lawson semilattice [CHK2, Th.2.4].

To prove the "if" part, assume that S is a compact topological inverse Clifford semigroup S with Lawson semilattice E of idempotents. By Corollary 1 of [Hr], S embeds into a product $\prod_{\alpha\in A}\hat{H}_\alpha$ of the cones over compact topological groups H_α. By definition, for a compact topological group G the semigroup

$$\hat{H} = H \times [0,1]/H \times \{0\}$$

that is the quotient semigroup of the product $H \times [0,1]$ of H with the min-interval $[0,1]$ by the ideal $H \times \{0\}$ of $H \times [0,1]$.

Observe that the unit interval $[0,1]$ is the image of the standard Cantor set $C \subset [0,1]$ under a continuous monotone map $h : C \to [0,1]$ well-known under the name "Cantor ladder". The map h can be thought as a continuous semilattice homomorphism $h : C \to [0,1]$, where both C and $[0,1]$ are endowed with the operation of minimum. Then $\hat{H}$ is the image of the semigroup $H \times C$, which is a compact topological inverse Clifford semigroup with zero-dimensional idempotent semilattice C.

Thus for each index $\alpha \in A$ with can construct a continuous surjective homomorphism $h_\alpha : S_\alpha \to \hat{H}_\alpha$ of a compact topological inverse Clifford semigroup S_α with zero-dimensional idempotent semilattice onto the semigroup $\hat{H}_\alpha$. Taking the product of those homomorphisms we obtain a continuous surjective homomorphism

$$h : \prod_{\alpha\in A} S_\alpha \to \prod_{\alpha\in A} \hat{H}_\alpha .$$

It is clear that $\prod_{\alpha\in A}S_\alpha$ is a compact topological inverse Clifford semigroup with zero-dimensional idempotent semilattice. By Theorem 3(2), this semigroup belongs to the class $\mathscr{H}_0$ and so does its subsemigroup $S_0 = h^{-1}(S)$. It remains to observe that S is the continuous homomorphism image of the semigroup $S_0 \in \mathscr{H}_0$. □

This proposition yields many examples of compact Clifford semigroups $S \notin \mathscr{H}$ that are continuous homomorphic images of compact Clifford semigroups $S_0 \in \mathscr{H}_0 \subset \mathscr{H}$. We have also a non-Clifford example.

Example 1. *The holomorph* $\mathrm{Hol}(E_3) = E_3 \rtimes Aut(E_3)$ *of the 3-element semilattice* $E_3 = \{e, f, ef\}$ belongs to the class $\mathscr{H}_0$ *but contains the 2-element ideal* $I = \{ef\} \times Aut(E_3)$ *such that quotient semigroup* $\mathrm{Hol}(E_3)/I$ *is isomorphic to the 5-element Brandt semigroup* $B(\mathbb{Z}_1, 2)$ *and thus does not belong to the class* $\mathscr{H}$.

The remaining part of the paper is devoted to the proofs of the results announced in the Introduction.

2. Semidirect products of topological semigroups

In this section we shall prove that the class $\mathscr{H}$ is closed under semidirect products with Abelian topological groups.

Let G be a topological group. By a topological *G-semigroup* we understand a topological semigroup S endowed with a homomorphism $\sigma \to G \to Aut(S)$ of G to the group of topological automorphisms of S such that the induced action $\tilde{\sigma} : G \times S \to S$, $\tilde{s} : (g, s) \mapsto \sigma(g)(s)$, is continuous. It will be convenient to denote the element $\sigma(g)(s)$ by the symbol gs.

The *semidirect product* $S \rtimes^{\sigma} G$ of a topological G-semigroup S with G is the topological semigroup whose underlying topological space is $S \times G$ and the semigroup operation is defined by $(s, g) * (s', g') = (s \cdot gs', gg')$. If the action σ of the group G on S is clear from the context, then we shall omit the symbol σ and will write $S \rtimes G$ instead of $S \rtimes^{\sigma} G$.

The following proposition describes some algebraic properties of semidirect products.

Proposition 3. *Let $S \rtimes G$ be the semidirect product of a topological G-semigroup S and a topological group G.*

(1) *$S \rtimes G$ is a (topological) inverse semigroup if and only if S is a (topological) inverse semigroup;*
(2) *$S \rtimes G$ is a topological group if and only if S is a topological group;*
(3) *$S \rtimes G$ is an inverse Clifford semigroup if and only if S is an inverse Clifford semigroup and $ge = e$ for any $g \in G$ and any idempotent e of S.*

Proof. First observe that S can be identified with the subsemigroup $S \times \{e\}$ of $S \rtimes G$ where e is the unique idempotent of G.

1. Assume that S is an inverse semigroup. To show that $S \rtimes G$ is a inverse semigroup we should check that the idempotents of $S \rtimes G$ commute and each element $(s, g) \in S \rtimes G$ has an inverse. For this observe that $(g^{-1}s^{\ 1}, g^{\ 1})$ is an inverse element to (s, g). Indeed,

$$(s, g) * (g^{\ 1}s^{\ 1}, g^{\ 1}) * (s, g) = (ss^{\ 1}, e)(s, g) = (ss^{-1}s, g) = (s, g).$$

By analogy we can check that

$$(g^{-1}s^{\ 1}, g^{-1})(s, g)(g^{-1}s^{\ 1}, g^{\ 1}) = (g^{\ 1}s^{\ 1}, g^{-1}).$$

Observe that an element (s, g) is an idempotent of the semigroup $S \rtimes G$ is and only if s and g are idempotents. This observation easily implies that the idempotents of the semigroup $S \rtimes G$ commute (because the idempotents of S commute).

If S is a topological inverse semigroup, then the map $(\cdot)^{-1} : S \to S, (\cdot)^{-1} : s \mapsto s^{-1}$ is continuous. The continuity of this map can be used to show that the map

$$(\cdot)^{-1} : S \rtimes G \to S \rtimes G, \qquad (\cdot)^{-1} : (s, g) \mapsto (g^{-1}s^{-1}, g^{-1})$$

is continuous too.

Next, assume that $S \rtimes G$ is an inverse semigroup. Given any element s consider the element $x = (s, e) \in S \rtimes G$ and find its inverse $x^{-1} = (s', g)$ in $S \rtimes G$. It follows from $(s, e)(s', g)(s, e) = xx^{-1}x = x = (s, e)$ that $g = e$ and then $ss's = s$ and $s'ss' = s$, which means that s' is the inverse element to s in the semigroup S. Since the idempotents of $S \rtimes G$ commute and lie in the subsemigroup $S \times \{e\}$, the idempotents of S commute too, which yields that S is an inverse semigroup.

If $S \rtimes G$ is a topological inverse semigroup, then S is a topological inverse semigroup, being a subsemigroup of $S \rtimes G$.

2. The second item follows from the first one and the fact that a topological semigroup is a topological group if and only if it is a topological inverse semigroup with a unique idempotent.

3. Assume that the semigroup S is inverse and Clifford, and G acts trivially on the idempotents of S. By the first item, $S \rtimes G$ is an inverse semigroup. So it remains to prove that $xx^{-1} = x^{-1}x$ for all $x = (s, g) \in S \rtimes G$. Observe that $x^{-1} = (g^{-1}s^{-1}, g^{-1})$ and thus

$$\begin{aligned} x^{-1}x &= (g^{-1}s^{-1}, g^{-1})(s, g) = (g^{-1}s^{-1}g^{-1}s, e) = (g^{-1}(s^{-1}s), e) = \\ &= (s^{-1}s, e) = (ss^{-1}, e) = (s, g)(g^{-1}s^{-1}, g^{-1}) = xx^{-1}. \end{aligned}$$

Here we used that G acts trivially on the idempotents of S and hence $g^{-1}(s^{-1}s) = s^{-1}s$. We also used that fact that $g^{-1} : s \mapsto g^{-1}s$ is an automorphism of the semigroup S and thus $g^{-1}(s^{-1}s) = (g^{-1}s^{-1})(g^{-1}s)$. Now assume that the semigroups $S \rtimes G$ is Clifford and inverse. The S is Clifford, being a subsemigroup of $S \rtimes G$. It remains to show that G acts trivially on the idempotents of S. Take any idempotent $s \in S$, any $g \in G$, and consider the element $x = (s, g)$ and its inverse $x^{-1} = (g^{-1}s^{-1}, g^{-1})$. Since $S \rtimes G$ is Clifford, $xx^{-1} = x^{-1}x$, which implies that

$$\begin{aligned} x^{-1}x &= (g^{-1}s^{-1}, g^{-1})(s, g) = (g^{-1}s^{-1}g^{-1}s, e) = (g^{-1}s^{-1}s, e) = \\ &= (g^{-1}s, e) = xx^{-1} = (ss^{-1}, e) = (s, e) \end{aligned}$$

and thus $gs = s$. $\square$

If S is a topological G-semigroup, then $\exp(S)$ has a structure of a topological G-semigroup with respect to the induced action

$$G \times \exp(S) \to \exp(S), \qquad (g, K) \mapsto gK = \{gs : s \in K\}.$$

Thus is it legal to consider the semidirect product $\exp(S) \rtimes G$.

The proof of the following proposition is easy and is left to the reader.

Lemma 1. *The map*

$$E : \exp(s) \rightthreetimes G \to \exp(S \rightthreetimes G), \qquad E : (K, g) \mapsto K \times \{g\}$$

is a topological embedding of the topological semigroups.

For a topological semigroup S consider the Tychonov power S^G as a topological G-semigroup with the following action of G:

$$\left(g, (s_\alpha)_{\alpha \in G}\right) \mapsto (s_{g\alpha})_{\alpha \in G}.$$

A homomorphism $h : S \to S'$ between two G-semigroups is called *G-equivariant* if $h(gs) = gh(s)$ for every $g \in G$ and $s \in S$. The proof of the following lemma also is left to the reader.

Lemma 2. *For any topological semigroup H the map*

$$E : \exp(H)^G \to \exp(H^G), \qquad E : (K_\alpha)_{\alpha \in G} \mapsto \prod_{\alpha \in G} K_\alpha$$

is a G-equivariant embedding of the corresponding G-semigroups.

The following immediate lemma helps to transform semigroup embedding into G-equivariant embedding.

Lemma 3. *Let G be an Abelian topological group. If $f : S \to H$ is an embedding of a topological G-semigroup H into a topological semigroup H, ten the map*

$$F : S \to H^G, \qquad F : s \mapsto (f(gs))_{g \in G}$$

is a G-equivariant embedding of the G-semigroup S into the G-semigroup H^G.

Finally we are able to prove the second item of Theorem 1.

Theorem 5. *Let G be an Abelian topological group. If a topological G-semigroup S embeds into the hypersemigroup* $\exp(H)$ *of a topological group H, then the semidirect product $S \rightthreetimes G$ embeds into the hypersemigroup* $\exp(H^G \rightthreetimes G)$ *of the topological group $H^G \rightthreetimes G$.*

Proof. Let $f : S \to \exp(H)$ be an embedding. By Lemmas 3 and 2, the map

$$F : S \to \exp(H^G), \qquad F : s \mapsto \prod_{\alpha \in G} f(\alpha s)$$

is a G-equivariant embedding. The G-equivariantness of F guarantees that the map

$$E : S \rightthreetimes G \to \exp(H^G) \rightthreetimes G, \qquad E : (s, g) \mapsto (F(s), g)$$

is an embedding of the corresponding topological semigroups. Finally, applying Lemma 1 we see that the semigroup $S \rightthreetimes G$ admits an embedding into the hypersemigroup $\exp(H^G \rightthreetimes G)$ of the topological group $H^G \rightthreetimes G$. $\square$

3. Idempotents and invertible elements of the hypersemigroups

In this section given a topological group G we characterize idempotent and related special elements of the hypersemigroup $\exp(G)$. We recall that an element x of a semigroup S is called

- an *idempotent* if $xx = x$;
- *regular* if there is an element $y \in S$ such that $xyx = x$;
- *(uniquely) invertible* if there is a (unique) element $x^{-1} \in S$ such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$;
- *a group element* if x lies in some subgroup of S.

It is possible to prove our results in a more general setting of cancellative topological semigroups. We recall that a semigroup S is *cancellative* if for any $x, y, z \in S$ the equality $xz = yz$ implies $x = y$ and the equality $zx = zy$ implies $x = y$. It is easy to check that the invertible elements of a cancellative semigroup form a subgroup.

Proposition 4. *Let X be a cancellative topological semigroup. A non-empty compact subset $K \subset X$ is*

(1) *an idempotent of the semigroup* $\exp(X)$ *if and only if K is a compact subgroup of X;*

(2) *a regular element of the semigroup* $\exp(X)$ *if and only if K uniquely invertible in* $\exp(X)$ *if and only if $K = Hx$ for some compact subgroup $H \subset X$ and some invertible element $x \in X$;*

(3) *a group element in* $\exp(X)$ *if and only if $K = Hx = xH$ for some compact subgroup $H \subset X$ and some invertible element $x \in X$.*

Proof. 1. If a compact subset $K \subset X$ is an idempotent of the semigroup $\exp(X)$ that is $KK = K$, then K is a compact cancellative semigroup. It is known [CHK1, Th. 1.10] that a compact cancellative semigroup is a group. If K is subgroup of X then $KK = K$.

2. Assume that $K \in \exp(X)$ is a regular element of the semigroup $\exp(X)$ which means that $KAK = K$ for some non-empty compact subset $A \subset X$. Fix any element $x \in K$ and $a \in A$. The set KA, being an idempotent of the semigroup $\exp(X)$, coincides with some compact subgroup H of X. We claim that $K = Hx$ and the element x is invertible in X. Observe that $Hx \subset HK = KAK = K$ and thus $Hxa \subset KA = H$, which implies that $xa = H$ is invertible. Consequently, $xa(xa)^{-1} = e = (xa)^{-1}xa$ which means that x and a are invertible. It follows from $Ka \subset KA = H$ that

$$K \subset Ha^{-1} = Ha^{-1}x^{-1}x = H(xa)^{-1}x \subset HHx = Hx \subset K$$

and thus $K = Hx$.

To show that K is uniquely invertible, assume additionally that $AKA = A$. In this case $A = AKA \supset aKa = aHxa = aH = x^{-1}xaH = x^{-1}H$. On the other hand, the equality $KAK = K$ implies $xAx \subset Hx$ and $A \subset x^{-1}H$. Therefore $A = x^{-1}H$ is a unique inverse element to K.

3. If $K = Hx = xH$ for some compact subgroup $H \subset X$ and some invertible element $x \in X$, then for the element $K^{-1} = x^{-1}H = Hx^{-1}$ we get $K^{-1}K = KK^{-1} = H$, which implies that K is a group element of $\exp(X)$. Conversely, if K is a group element, then $KK^{-1} = K^{-1}K = H$ for some compact subgroup $H \subset X$ and $K = Hx$ for some invertible element $x \in X$ (because K is regular). Since $H = K^{-1}K = x^{-1}HHx = x^{-1}Hx$, we get $xH = Hx$. □

Theorem 2 is particular case of the following more general

Theorem 6. *Let X be a cancellative topological semigroup and G be the subgroup of invertible elements of X. Let S be an algebraically regular subsemigroup of $\exp(X)$ and E be the set of idempotents of S.*

(1) *The semigroup S is inverse and $S \subset \exp(G)$.*
(2) *If G is a topological group, then S is a topological inverse semigroup.*
(3) *An element $x \in S$ is an idempotent if x^2x^{-1} is an idempotent.*
(4) *Any distinct conjugate idempotents of S are incomparable.*
(5) *The set E is a closed commutative subsemigroup of S and for every $e \in E$ the upper cone $\uparrow e = \{f \in E : ef = e\}$ is totally disconnected.*

Proof. 1. Let S be a regular subsemigroup of $\exp(X)$. It follows from Proposition 4 that each element $K \in S$, being regular, is equal to Hx for some compact subgroup $H \subset G$ and some invertible element $x \in X$. Then $K = Hx \subset G$ and hence $K \in \exp(G) \subset \exp(X)$. By Proposition 4, K is uniquely invertible in $\exp(X)$ and hence in S, which means that S is an inverse semigroup. Moreover, the inverse K^{-1} to K in S can be found by the natural formula: $K^{-1} = \{x^{-1} : x \in K\}$.

2. If the subgroup G of invertible elements of X is a topological group, then the inversion

$$(\cdot)^{-1} : \exp(G) \to \exp(G), \qquad (\cdot)^{-1} : K \mapsto K^{-1}$$

is continuous with respect to the Vietoris topology on $\exp(G)$ and consequently, the inversion map of S is continuous as well, which yields that S is a topological inverse semigroup.

3. Let $K \in S$ be an element such that K^2K^{-1} is an idempotent in S and hence is a compact subgroup of X. By Proposition 4, $K = Hx$ for some compact subgroup H of X and some invertible element $x \in X$. Then $K^2K^{-1} = HxHxx^{-1}H = HxH$. The set K^2K^{-1}, being a subgroup of X, contains the neutral element 1 of X. Then $1 \in K^2K^{-1} = HxH$ and hence $x \in H$, which implies that $K = Hx = H$ is an idempotent in $\exp(X)$ and S.

4. Let E, F be two distinct conjugate idempotents of the semigroup S. Find an element $K \in S$ such that $E = KFK^{-1}$ and $F = K^{-1}EK$. By Proposition 4, find

a compact subgroup H of X and an invertible element $x \in X$ such that $K = Hx$. We claim that $E = xFx^{-1}$. Indeed, the inclusion

$$x^{\ 1}Hx = x^{\ 1}HHx = K^{\ 1}K \subset K^{-1}EK = F$$

implies

$$E = KFK^{\ 1} = HxFx^{\ 1}H = xx^{-1}HxFx^{\ 1}Hxx^{\ 1} \subset xFFFx^{\ 1} = xFx^{\ 1}.$$

On the other hand,

$$H = Hxx^{\ 1}H \subset HxFx^{\ 1}H = KFK^{\ 1} = E$$

implies

$$F = K^{-1}EK = x^{-1}HEHx \subset x^{-1}EEEx = x^{-1}Ex$$

and hence $xFx^{-1} \subset E$.

5. Since S is an inverse semigroup, the set E of idempotents of S is a commutative subsemigroup of S, see [Pet, II.1.2]. To show that E is closed in S, pick any element $K \in S \setminus E$. By Proposition 4, $K = Hx$ for a compact subgroup $H \subset X$ and an invertible element $x \in X$. Since K is not an idempotent, Hx is not a subgroup, which means that the neutral element 1 of H does not belong to Hx. Let $U = X \setminus \{x\}$ and observe that $U^+ = \{C \in \exp(X) : 1 \notin C\}$ is a neighborhood of K in $\exp(X)$ that contains no subgroup of X and hence does not intersect the set E.

Now given an idempotent $H \in \mathscr{E}$ we shall prove that the upper cone $\uparrow H = \{E \in \mathscr{E} : HE = H\}$ of H is totally disconnected. By Proposition 4, H is a compact subgroup of X. It follows that $\uparrow H \subset \exp(H)$. The total disconnectedness of $\uparrow H$ will be proven as soon as given two distinct elements $E_0, E_1 \in \uparrow H$ we find a closed-and-open subset $\mathscr{U} \subset \uparrow H$ such that $E_0 \in \mathscr{U}$ but $E_1 \notin \mathscr{U}$. We loose no generality assuming that $X = H$ and hence $\mathscr{E} = \uparrow H \subset \exp(H)$.

We first consider the special case when H is a Lie group. Without loss of generality $E_0 \not\subset E_1$ and hence $E_0 \notin \downarrow E_1 = \{E \in \mathscr{E} : E \subset E_1\}$. So, it remains to prove that the lower cone $\downarrow E_1$ is closed-and open in $\mathscr{E}$. The closedness of $\downarrow E_1$ follows from the continuity of the semigroup operation and the equality $\downarrow E_1 = \{E \in \mathscr{E} : EE_1 = E_1\}$. To prove that $\downarrow E_1$ is open in $\mathscr{E}$, take any $K \in \downarrow E_1$. The set $K \in \exp(H)$, being an idempotent of the semigroup $\mathscr{E}$ is a closed subgroup of H.

By Corollary II.5.6 of [Bre] the subgroup K of the compact Lie group H has an open neighborhood $O(K) \subset H$ such that for each compact subgroup $C \subset O(K)$ satisfies the inclusion $xCx^{-1} \subset K$ for a suitable point $x \in H$. We shall derive from this that $C = K$ provided $C \supset K$. Indeed, $C \supset K$ and $xCx^{\ 1} \subset K$ imply $xKx^{-1} \subset xCx^{-1} \subset K$. Being a homeomorphic copy of the group K, the subgroup $xKx^{\ 1} \subset K$ must coincide with K (it has the same dimension and the same number of connected components). Consequently, $xCx^{\ 1} = K$ and hence C, being homeomorphic to its subgroup K, coincides with K too.

The continuity of the semigroup operation of $\mathscr{E}$ yields a neighborhood $O_1(K) \subset \mathscr{E}$ of K such that $EK \subset O(K)$ for each $E \in O_1(K)$. We claim that $O_1(K) \subset \downarrow E_1$. Take any element $E \in O_1(K)$ and observe that the product EK, being an idempotent in the semigroup $\mathscr{E}$, is a compact subgroup of H containing the subgroup K. Now the choice of the neighborhood $O(K)$ guarantees that $E \subset EK \subset K \subset E_1$ and hence $E \subset \downarrow E_1$. This proves that $O_1(K) \subset \downarrow E_1$, witnessing that $\downarrow E_1$ is open in $\mathscr{E}$.

Now we are able to finish the proof assuming that H is an arbitrary compact topological group. Given distinct elements $E_0, E_1 \in \mathscr{E} \subset \exp(H)$ we should find an closed-and-open subset $\mathscr{U} \subset \mathscr{E}$ containing E_0 but not E_1. The topological group H, being compact, is the limit of an inverse spectrum consisting of compact Lie group. Consequently, we can find a continuous homomorphism $h: H \to L$ onto a compact Lie group L such that $h(E_0)$ and $h(E_1)$ are distinct subgroups of L. It follows that $h(\mathscr{E}) = \{h(E): E \in \mathscr{E}\}$ is an idempotent semigroup of the hypersemigroup $\exp(L)$. Now the particular case considered above yields a closed-and-open subset $\mathscr{V} \subset h(\mathscr{E})$ containing $h(E_0)$ but not $h(E_1)$. By the continuity of the homomorphism h the set $\mathscr{U} = \{K \in \mathscr{E} : h(K) \in \mathscr{V}\}$ is closed-and-open in $\mathscr{E}$. It contains E_0 but not E_1. This proved the total disconnectedness of the upper cone $\uparrow H$. $\square$

4. Proof of theorem 4

In this section we will prove the Theorem 4. Given a Clifford compact topological semigroup S and a non-trivial variety $\mathscr{G}$ of compact topological groups we should prove that S embeds into the hypersemigroup $\exp(G)$ of a topological group $G \in \mathscr{G}$ if and only if S is a topological inverse semigroup whose idempotent semilattice E is zero-dimensional and all maximal groups H_e, $e \in E$, belong to the class $\mathscr{G}$.

To prove the "if" part, assume that S is a compact Clifford topological inverse semigroup whose idempotent semilattice E is zero-dimensional and all maximal groups H_e, $e \in E$, belong to the class $\mathscr{G}$. For every $e \in E$ let $\tilde{H}_e = E_e$ if H_e is not trivial and $\tilde{H}_e \in \mathscr{G}$ be any non-trivial compact group if H_e is trivial (such a group $\tilde{H}_e$ exists because the variety $\mathscr{G}$ is not trivial). Since $\mathscr{G}$ is closed under taking Tychonov products, the compact topological group $G = \prod_{e \in E} \hat{H}_e$ belongs to $\mathscr{G}$. Finally, by Theorem 3(5), the semigroup S embeds into $\exp(G)$.

To prove the "only if" part, assume that S embeds into the hypersemigroup $\exp(G)$ over a topological group $G \in \mathscr{G}$. By Theorem 3(3), S is a compact topological inverse Clifford semigroup with zero-dimensional idempotent semilattice E. It remains to show that each maximal group H_e, $e \in E$, of S belongs to $\mathscr{G}$. The embedding of S into $\exp(G)$ induces an embedding $h: H_e \to \exp(G)$. The image $H_0 = h(e)$, being an idempotent in $\exp(G)$, is a compact subsemigroup of G and thus a compact subgroup of G according to Theorem 1.10 [CHK1]. The same is true for the semigroup $H = \bigcup_{x \in H_e} h(x)$. It is a compact subgroup of G. We claim that H_0 is a normal subgroup of H.

Indeed, for any $x \in H$ we can find a point $z \in H_e$ with $x \in h_e(z)$. It follows from (the proof of) Proposition 4(3) that $h_e(z) = xH_0 = H_0 x x H_0 x^{-1} = H_0$, witnessing that the subgroup H_0 is normal in H.

Let $\pi : H \to H/H_0$ be the quotient homomorphism. It follows from Proposition 4(3) that the composition $\pi \circ h_e : H_e \to H/H_0$ is a bijective continuous homomorphism. Because of the compactness of H_e, the group H_e is isomorphic to $H\ H_0$, which, being the quotient group of the closed subgroup H of the group $G \in \mathcal{G}$ belongs to the variety $\mathcal{G}$.

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