Paul H. Rabinowitz

A priori bounds for a semilinear wave equation

In: Jiří Fábera (ed.): Equadiff IV, Czechoslovak Conference on Differential Equations and Their Applications. Proceedings, Prague, August 22-26, 1977. Springer-Verlag, Berlin, 1979. Lecture Notes in Mathematics, 703. pp. [340]--347.

Persistent URL: http://dml.cz/dmlcz/702234

A priori bounds for a semilinear wave equation

Paul H. Rabinowitz

The purpose of this note is to describe some recent results on the existence and regularity of solutions of semilinear wave equations of the form

$$\text{(1)} \qquad \begin{cases} u_{tt} - u_{xx} + f(u) = 0 , & 0 < x < \pi , \quad t \in \mathbb{R} \\ u(0,t) = 0 = u(\pi,t) \end{cases}$$

where $f(0) = 0$. Of interest is the existence of time periodic solutions of (1) . Note that $u \equiv 0$ is a trivial such solution; nontrivial solutions are often called free vibrations for (1) . The same methods we shall describe below can be used to treat the forced vibration case where f also depends explicitly on t in a time periodic fashion.

There is a substantial literature on forced and free vibration problems for (1) , mainly for the former case with f being a perturbation term, i.e. $f = \varepsilon g(x,t,u)$ where ε is small. See e.g. [1-8] and the references cited there. The work to be discussed here can be found in detail in [9].

Our main result for (1) is:

<u>Theorem 2</u>: Suppose $f \in C^k$, $k \geq 2$, and satisfies

(f1) f is strictly monotone increasing with $f(0) = 0$,

(f2) f is superlinear at 0 and ∞ , i.e.

(i) $f(z) = o(|z|)$ at $z = 0$

(ii) $F(z) = \int_0^z f(s)ds \leq \theta z f(z)$ for $|z| \geq \overline{z}$ where $\theta \in [0,\frac{1}{2})$.

Then for any period T which is a rational multiple of π , (1) possesses a nontrivial T-periodic solution $u \in C^k$.

$$\|u\|_E^2 = \iint_D (u_t^2 + u_x^2)\,dx\,dt$$

where $D = [0,\pi] \times [0,2\pi]$. Ignoring questions of where it is defined, formally critical points of

$$(3) \qquad \iint_D [\tfrac{1}{2}(u_t^2 - u_x^2) - F(u)]\,dx\,dt$$

in E are weak solutions of (1) . Unfortunately we know of no direct way of determining nontrivial critical points of (3) . However if a finite dimensional approximation argument is used, i.e. (3) is restricted to $E_m = \text{span}\{\sin jx \ \sin kt \ , \ \sin jx \ \cos kt \mid 0 \le j,k \le m\}$, the form of (3) and hypotheses on f imply the existence of a nontrivial critical point. The problem then becomes that of finding bounds for this critical point

Before sketching the proof, a few remarks are in order. We do not know if the restriction on T is essential. The reason for this assumption here is that for such t , the spectrum of

$\Box = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}$ in the class of T periodic functions in t satisfying the boundary conditions in x is discrete with 0 being an isolated point in the spectrum while if T is irrational, the spectrum is dense and 0 is an accumulation point of the spectrum. Condition (f2)(i) can be eliminated provided that (fl) is retained. We do not know if the monotonicity of f can be relaxed in any essential way. See however [7] . Lastly condition (f2)(ii) implies $|f(z)| \ge a_1 |z|^{\frac{1}{\theta}-1} - a_2$; hence the terminology superlinearity at ∞ .

Turning now to the proof of Theorem 2 , the basic idea is to try to find a solution of (1) as a critical point of a corresponding functional. For convenience we take $T = 2\pi$. Consider the set of smooth functions 2π periodic in t and having compact support (in $(0,\pi)$) in x . Let E denote the closure of this set with respect to which enable us to pass to a limit and get a nontrivial solution of (1). Two technical problems impede this program. To describe them, consider the linear problem

$$(4) \qquad \begin{cases} \Box w = g(x,t) , \\ w(0,t) = 0 = w(\pi,t) ; \; w(x,t+2\pi) = w(x,t) . \end{cases}$$

Under these boundary and periodicity conditions, $\Box$ has a null space whose closure in $L^2(T) \equiv L^2$ is $N = \{p(x+t) - p(-x+t) \mid p \in L^2(S^1)\}$. If $N^\perp$ denotes the orthogonal complement of N in L^2, then for $g \in N^\perp$, (4) is uniquely invertible in $N^\perp$ with a gain of one derivative in either the L^2 or sup norms [4,5]. Thus we have some compactness for the projection of (1) in $N^\perp$; however there is none in N.

The two technical difficulties mentioned above are: (i) the unrestricted growth of f at ∞ does not permit us to obtain the necessary estimates for w, the component of u in $N^\perp$; (ii) the lack of compactness of the projection of (1) in N.

To get around these difficulties, we modify (1) and (3). For $u \in E$, we can write $u = v + w$ where $v \in N$ and $w \in N^\perp$. Let $\beta > 0$. Consider

$$(5) \qquad \begin{cases} \Box u - \beta v_{tt} + f_K(u) = 0 , \quad 0 < x < \pi , \quad t \in \mathbb{R} \\ u(0,t) = 0 = u(\pi,t) ; \; u(x,t+2\pi) = u(x,t) \end{cases}$$

where f_K satisfies (f1)-(f2), $f_K(z) = f(z)$ for $|z| \le K$, and f_K grows at a prescribed rate, e.g. cubically, at ∞. The β term essentially compactifies the projection of (1) on N. Corresponding to (5) we have the functional:

$$(6) \qquad I(u) = \iint_D \left[\tfrac{1}{2}(u_t^2 - u_x^2 - \beta v_t^2) - F_K(u)\right] dx\, dt$$

where F_K is the primitive of f_K. The idea now is to: 1^o find an appropriate critical point of $I|_{E_m}$; 2^o get suitable estimates for this critical point; 3^o pass to a limit and solve (5); 4^o get β and K independent estimates for solutions of (5); and 5^o let $\beta \to 0$ and $K \to \infty$ to solve (1). This is too lengthy a process for us to carry out now so we will content ourselves with just trying to give the flavor of a few of the estimates that are involved. To do this we return to (1) and argue <u>a priori</u>. This is much simpler than the actual procedure carried out in the existence argument.

Thus suppose we have a smooth solution u, of (1). We will obtain bounds for u in terms of c, the critical value of I corresponding to u. Thus suppose $I(u) = c$. The first estimate gives a bound for $\|f(u)\|_{L^1}$. Since $I'(u) = 0$ (where $I'(u)$ denotes the Frechet derivative of I at u),

$$c = I(u) - \frac{1}{2} I'(u)u = \iint_D [\frac{1}{2} f(u)u - F(u)]\, dx\, dt \tag{7}$$

Invoking (f2)(ii) then gives

$$\|f(u)u\|_{L^1} \le M_1 \tag{8}$$

for some constant M_1 depending on c. By (f1), $|f(z)| \le f(1) - f(-1) + f(z)z$. Hence (8) implies a bound for $\|f(u)\|_{L^1}$.

Next writing $u = v + w$, $v \in N$, $w \in N^{\perp}$, we have

$$\Box w = -f(w). \tag{9}$$

There is a representation theorem [5] for solutions of (4) which implies:

$$\|w\|_{L^\infty} \le a_3 \|g\|_{L^1}. \tag{10}$$

Consequently we conclude

(11) $$\|w\|_{L^\infty} \le a_3 \|f(u)\|_{L^1} \le a_3 M_1 \equiv M_2 \ .$$

The next step is to get an estimate for $\|v\|_{L^\infty}$. This is more subtle. We assume $v \not\equiv 0$ or there is nothing to prove. From (9) we conclude that

(12) $$\iint_D f(u)\varphi \, dx \, dt = 0$$

for all $\varphi \in N$. By choosing φ to be an appropriate nonlinear function of v , we will obtain the desired estimate for $\|v\|_{L^\infty}$. By rewriting (12) we get

(13) $$\iint_D (f(v+w) - f(w))\varphi \, dx \, dt = - \iint_D f(w)\varphi \, dx \, dt$$
$$\le \|f(w)\|_{L^\infty} \iint_D |\varphi| \, dx \, dt$$

From the definition of N we have $\varphi = p(x+t) - p(-x+t)$ where $p \in L^2(S^1)$. Clearly p is only determined up to an additive constant. We make p unique by requiring that

$$[p] \equiv \int_0^{2\pi} p(s)ds = 0 \ .$$

Define a function $q(s)$ by $q(s) = 0$ if $|s| \le M$, $q(s) = s - M$ if $s > M$; $q(s) = s + M$ if $s < -M$. With the above normalization on p , we write $v(x,t) = p(x+t) - p(-x+t) \equiv v^+ - v^-$ and chose $\varphi = q(v^+) - q(v^-) \equiv q^+ - q^- \in N$. Therefore for any $\delta > 0$, by (f1) ,

(14) $$\iint_{D_\delta} (f(v+w) - f(w))(q^+ - q^-) \, dx \, dt \le \|f(w)\|_{L^\infty} (\|q^+\|_{L^1} + \|q^-\|_{L^1})$$

where $D_\delta = \{(x,t) \in D \,\big|\, |v| \geq \delta\}$. Let $D^+ = \{(x,t) \in D_\delta | v \geq \delta\}$ and $D^- = D_\delta \backslash D^+$. Define

$$\psi(z) = \begin{cases} \min\limits_{|\zeta| \leq M_2} f(z+\zeta) - f(\zeta) & z \geq 0 \\ \max\limits_{|\zeta| \leq M_2} f(z+\zeta) - f(\zeta) & z < 0 \end{cases}$$

Then by (fl) , ψ is strictly monotone increasing and $|\psi(z)| \to \infty$ as $|z| \to \infty$. From the definition of ψ we get

$$(15) \qquad \iint_{D^+} (f(v+w) - f(w))(q^+ - q^-)\,dx\,dt \geq$$

$$\geq \iint_{D^+} \frac{\psi(v)}{v} v(q^+ - q^-)\,dx\,dt \geq \frac{\psi(\delta)}{\|v\|_{L^\infty}} \iint_{D^+} v(q^+ - q^-)\,dx\,dt$$

since $v(q^+ - q^-) \geq 0$. Similar estimates for the T^- integral lead to

$$(16) \qquad \iint_{D_\delta} (f(v+w) - f(w))(q^+ - q^-)\,dx\,dt \geq \frac{\mu(\delta)}{\|v\|_{L^\infty}} \iint_{D_\delta} v(q^+ - q^-)\,dx\,dt$$

where for $z \geq 0$, $\mu(z) = \min(\psi(z), -\psi(-z))$. Note that μ is strictly monotone increasing and $\mu(z) \to \infty$ as $z \to \infty$. Now

$$(17) \qquad \iint_{D_\delta} v(q^+ - q^-)\,dx\,dt \geq \iint_D v(q^+ - q^-)\,dx\,dt - \delta(\|q^+\|_{L^1} + \|q^-\|_{L^1}) .$$

Since $[v^\pm] = 0$, it is easy to verify that

$$\iint_D v^+ q^- \,dx\,dt = 0 = \iint_D v^- q^+ \,dx\,dt .$$

Therefore

$$(18) \qquad \iint_T v(q^+ - q^-)\,dx\,dt = \iint_T (v^+ q^+ + v^- q^-)\,dx\,dt .$$

By the definition of q, we have $sq(s) \geq M|q(s)|$. Hence

$$\iint_D (v^+q^+ + v^-q^-)dx\,dt \geq M(\|q^+\|_{L^1} + \|q^-\|_{L^1}) . \tag{19}$$

Combining (14), (16)-(19) yields

$$\frac{(M-\delta)}{\|v\|_{L^\infty}} \mu(\delta)\,(\|q^+\|_{L^1} + \|q^-\|_{L^1}) \leq \|f(w)\|_{L^1}(\|q^+\|_{L^1} + \|q^-\|_{L^1}) . \tag{20}$$

Choosing any $M < \|v^+\|_{L^\infty} = \|v^-\|_{L^\infty}$, the L^1 terms are positive so they can be cancelled and

$$\frac{(M-\delta)}{\|v\|_{L^\infty}} \mu(\delta) \leq \|f(w)\|_{L^\infty} . \tag{21}$$

Since this is true for all $M < \|v^\pm\|_{L^\infty}$, we can take $M = \|v^\pm\|_{L^\infty}$. Further noting that $\|v\|_{L^\infty} \leq 2\|v^\pm\|_{L^\infty}$ and taking $\delta = \frac{1}{2}\|v\|^\pm_{L^\infty}$ yields

$$\|v^\pm\|_{L^\infty} \leq 2\mu^{-1}(4\|f(w)\|_{L^\infty}) . \tag{22}$$

Thus (22) and our estimate for $\|w\|_{L^\infty}$ give the desired bound for $\|v\|_{L^\infty}$. Therefore we have a bound for $\|u\|_{L^\infty}$.

To get further estimates, from (9) and the properties of $\Box^{-1}$ we have

$$\|w\|_{C^1} \leq a_4\|f(w)\|_{L^\infty} \leq M_3 . \tag{23}$$

Next the arguments used to obtain the bound for $\|v\|_{L^\infty}$ can be modified to estimate the modulus of continuity of v. In the framework of (5), these bounds enable us to pass to a limit to get a continuous weak solution of (1). A separate argument shows $u \not\equiv 0$. To verify that u is indeed a smooth solution of (1) requires further arguments which we will not carry out here.

References

[1] Vejvoda, O., Periodic solutions of a linear and a weakly nonlinear wave equation in one dimension, I, Czech. Math. J. 14, (1964), 341-382.

[2] Vejvoda, O., Periodic solutions of nonlinear partial differential equations of evolution, Proc. Sym. on Diff. Eq. and Applic. Brataslava—1966, (1969), 293-300.

[3] Kurzweil, J., Van der Pol perturbation of the equation for a vibrating string, Czech. Math. J., 17, (1967), 558-608.

[4] Rabinowitz, P. H., Periodic solutions of nonlinear hyperbolic partial differential equations, Comm. Pure Appl. Math., 20, (1967), 145-205.

[5] Lovicarová, H., Periodic solutions of a weakly nonlinear wave equation in one dimension, Czech. Math. J., 19, (1969), 324-342.

[6] Rabinowitz, P. H., Time periodic solutions of a nonlinear wave equation, Manus. Math., 5, (1971), 165-194.

[7] Štědrý, M. and O. Vejvoda, Periodic solutions to weakly nonlinear autonomous wave equations, Czech. Math. J., 25, (1975), 536-555.

[8] Brezis, H. and L. Nirenberg, Forced vibrations for a nonlinear wave equation, to appear.

[9] Rabinowitz, P. H., Free vibrations for a semilinear wave equation, to appear Comm. Pure Appl. Math.

This research was sponsored in part by the Office of Naval Research under Contract No. N00014-76-C-0300 and in part by the U.S. Army under Contract No. DAAG29-75-C-0024. Any reproduction in part or in full is permitted for the purposes of the U.S. Government.

Author's address: Department of Mathematics University of Wisconsin-Madison, Madison, Wis.53706, USA