ON THE CONVERGENCE OF SOLUTIONS OF RANDOM DIFFERENTIAL EQUATIONS

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We shall be dealing with the mean-square convergence of solutions of random differential equations.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $L_2(\Omega, \mathcal{F}, P)$ be a Banach space of all n-dimensional, real valued, second order random variables. The norm of $X \in L_2(\Omega, \mathcal{F}, P)$ is defined by

$$\|X\| = \max_i \left\{ \left[E|X_i|^2 \right]^{\frac{1}{2}} \right\},$$

where X_i $(i = 1, \dots, n)$ are projections of X on the one-dimensional subspaces and E denotes the operator of expectation.

The space of all linear bounded mappings from L_2 into L_2 is isomorphic with the space $\mathfrak{M}$ of all $n \times n$-matrices with the norm

$$|A| = \max_{1 \le i,j \le n} \left\{ |a_{ij}| \right\},$$

where a_{ij} $(i,j = 1, \dots, n)$ are real valued elements of the matrix $A \in \mathfrak{M}$.

The mapping

$$X : J \times \Omega \longrightarrow R^n, \qquad J = [t_0, \infty),$$

such that for arbitrary $t \in J$

$$X_t = X(t, \cdot) \in L_2(\Omega, \mathcal{F}, P),$$

i.e. X is the second order stochastic process, will be denoted by $\{X_t, t \in J\}$ or briefly by $\{X_t\}$ and the space of all mean-square continuous (resp. absolutely continuous) n-dimensional stochastic processes will be denoted by $C(J, L_2)$ (resp. $AC(J, L_2)$).

In this note we will consider asymptotic relation between solutions of the random differential equation of the type

$$(1) \quad \dot{X}_t = F(t, X_t, W_t), \qquad t \in J,$$

where $F : J \times L_2 \times L_2 \longrightarrow L_2$,

$\{X_t\} \in AC(J, L_2)$, $\{W_t\} \in C(J, L_2)$ and $\{\dot{X}_t\}$ is the mean-square derivative of the process $\{X_t\}$.

Here the conditions ensuring the existence of solutions are omitted (on this problem see for instance [1]), but nothing is said concerning the uniqueness of solutions.

Definition 1. It will be say that the stochastic process $\{X_t,\ t \in J\}$ is a S-solution of the random differential equation (1) if

(i) $J = [t_0, \infty)$,

(ii) $\{X_t,\ t \in J\} \in AC(J, L_2)$,

(iii) $\{X_t,\ t \in J\}$ satisfies condition (1) in the mean-square sense.

The set of all S-solutions of (1) will be denoted by S.

Definition 2. It will be say that all S-solutions of (1) are mean-square convergent to the common limit in infinity (briefly mean-square convergent) if for arbitrary $\{X_t\}, \{Y_t\} \in S$

$$\text{l.i.m. } X_t = \text{l.i.m. } Y_t, \text{ when } t \to \infty,$$

i.e.

$$\lim \| X_t - Y_t \| = 0.$$

First we consider the linear random differential equation

$$(2)\quad \dot{X}_t = A(t)\, X_t + W_t, \qquad t \in J,$$

where $\{W_t\} \in C(J, L_2)$ and $A(t)$ is a deterministic matrix.

Let $Q(t)$ be the corresponding to $A(t)$ fundamental matrix, i.e. the solution of homogeneous matrix differential equation

$$(3)\quad \dot{Q}(t) = A(t)\, Q(t), \qquad t \in J,$$

satisfying initial condition

$$Q(t_0) = I$$

(I denotes the unit matrix of order n).

Theorem 1. If the fundamental matrix $Q(t)$ satisfies condition

$$\lim | Q(t) | = 0,$$

then all S-solutions of random differential equation (2) are mean-square convergent.

If the matrix A is constant then the Eigen-values of A play an essential role. Namely, we have following Corollary from Theorem 1.

Corollary. If the Eigen-values of the matrix A have negative real parts, then all S-solutions of the random differential equation

$$(4)\quad \dot{X}_t = A\, X_t + W_t, \qquad t \in J,$$

are mean-square convergent.

The following Lemma (cf [2] and [4]) plays an important role in the proof of Theorem 2.

Lemma. If the function

$$\Phi : J \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

is differentiable and if x and y are solutions of deterministic differential equation

(5) $\dot{x} = \Phi(t, x), \qquad t \in J,\ x \in \mathbb{R}^n,$

then the difference

$$z(t) = x(t) - y(t), \qquad t \in J,$$

is a solution of quasi-linear differential equation

(6) $\dot{z}(t) = \Psi(t, x(t), y(t))\, z(t), \qquad t \in J,$

where

(7) $\Psi(t, x, y) = \int_0^1 \frac{\partial \Phi}{\partial x}(t, sx + (1 - s)y)\, ds$

and

$$\frac{\partial \Phi}{\partial x} = \left[\frac{\partial \Phi_i}{\partial x_j}\right]_{i,j=1,\dots,n.}$$

<u>Theorem 2</u>. If for arbitrary $\{U_t\}, \{V_t\} \in AC(J, L_2)$

$$\text{l.i.m. } \exp \int_{t_0}^{u} \Psi(t, U_t, V_t)dt = 0, \quad \text{as } u \longrightarrow \infty,$$

then all S-solutions of the random differential equation

(8) $\dot{X}_t = \Phi(t, X_t), \qquad t \in J,$

are mean-square convergent.

<u>Theorem 3</u>. If

(α) all solutions of the non-random differential equation

(9) $\dot{u}(t) = A(t)\, u(t), \qquad t \in J,$

are bounded,

(β) the function

$$\Phi : J \times L_2 \longrightarrow L_2$$

satisfies condition

(10) $\|\Phi(t, U) - \Phi(t, V)\| \le f(\|U - V\|), \quad U, V \in L_2,$

where the function

$$f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$$

is continuous and nondecreasing,

(γ) there exist a constant $H > 0$ and the function

$$\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$$

defined by

$$\int_H^u \frac{ds}{\varphi(s)} = \Psi(u), \qquad u \geq H,$$

and such that

$$\int_{t_0}^{t} \| Q^{-1}(s) \| \, ds = \lim_{u \to \infty} \Psi(u),$$

where Q^{-1} is the inverse matrix to the fundamental matrix in equation (9),

(δ) $\lim_{t \to \infty} \Psi^{-1}\left\{ \int_{t_0}^{t} \| Q^{-1}(s) \| \, ds \right\} = 0,$

(ε) $\{W_t\} \in C(J, L_2),$

then all S-solutions of the random differential equation

(11) $\dot{X}_t = A(t) X_t + \Phi(t, X_t) + W_t, \qquad t \in J,$

are mean-square convergent.

The proof of Theorem 3 is based on the lemma due to Bihari [1].

Remark 1. In general, sufficient conditions for mean-square convergence are weaker as conditions assuring asymptotic stability.

Remark 2. The mean-square convergence of all S-solutions of random differential equation implies uniqueness (in the sense of equivalence) of periodic or almost periodic solutions.

Remark 3. In the cases when the common limit exists only for any subset (resp. subsets) of solutions it should be interesting to study the properties of such subset, if it is open or closed, bounded or not and so on.

References

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