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On regularity of solutions to some problems of mathematical physics


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Let $\Omega \subset \mathbb{R}^m (m \geq 2)$ be a bounded domain with $C^1$ boundary $\partial \Omega$. Consider a boundary value problem for the elliptic system

\begin{equation}
(1) \quad Lu \equiv D_1 a_1(x, Du) = 0, \quad x \in \Omega,
\end{equation}

\begin{equation}
(2) \quad u(x) = g(x), \quad x \in \partial \Omega, \quad g \in W_2^1(\Omega).
\end{equation}

Here $u(x) = \{u_1^1(x), \ldots, u^N(x)\}$ is an unknown vector function, $Du = \{D_0u, D_1u, \ldots, D_mu\}$, $D_0u = u$, $D_1 = \frac{\partial}{\partial x_1}$ ($i = 1, \ldots, m$).

Until now we considered mostly the case when the coefficients $a_1(x, p) = \{a_1^1(x, p), \ldots, a_1^N(x, p)\}$ were measurable with respect to $x$, continuously differentiable with respect to $p = \{p_0, \ldots, p_m\}$ and for the matrix

\begin{equation}
(3) \quad A = \frac{\partial a_1}{\partial p_k}, \quad (i, k = 0, \ldots, m)
\end{equation}

the inequalities

\begin{equation}
(4) \quad \|A\| \|\xi\| \leq \mu \|\xi\| \leq \nu \|\xi\| \leq 2, \quad \|A\| \leq C
\end{equation}

took place with some positive constants $\mu$, $\nu$ and $C$. Suppose that $A$ is symmetric. Denote as $\{\lambda_i(x, p)\}$ the set of all eigenvalues of $A$ at the point $\{x, p\}$ and put $\Lambda = \inf \lambda_i$ and $\Lambda = \sup \lambda_i$. It was proved in [1] that if the inequality

\begin{equation}
(5) \quad \frac{\Lambda - \lambda_i}{\Lambda + \lambda_i} \left[ 1 + \frac{(m-2)^2}{m-1} \right]^{1/2} < 1
\end{equation}

holds then the weak solution $u$ of (1), (2) is locally H"{o}lder-continuous in $\Omega$. It was also proved that the condition (5) is sharp.

In some important problems the differentiability of the coefficients $a_1$ with respect to $p$ doesn't hold. It happens e.g. in case of the systems of equations for elasto-plastic media with hardening, the Maxwell systems for materials with ferromagnetic intrudings etc.

Suppose now that, instead of (4), the inequalities

\begin{equation}
(6) \quad \left( a(x, p) - a(x, q) \right) (p - q) \leq \mu \|p - q\|\quad \|a(x, p) - a(x, q)\| \leq \nu \|p - q\|
\end{equation}

take place with some positive constants $\mu$, $\nu$ for almost all $x \in \Omega$ and all $p, q \in \mathbb{R}^{(m+1)N}$. (The norms are calculated in $\mathbb{R}^{(m+1)N}$.)
Denote

(7) \[ K = \inf_{\varepsilon > 0} \sup_{\|p - q\|} \|p - q - \varepsilon [a(x, p) - a(x, q)]\| \] .

**THEOREM 1.** If

(8) \[ K \left[ 1 + \left( \frac{m-2}{m-1} \right)^2 \right]^{1/2} < 1, \]

then \( u \) is locally H"older continuous on \( \Omega \). The condition (8) is sharp.

Consider now the equilibrium system of the theory of small deformations for a material with strength condition (so called Hencky theory). Let \( S = \{\sigma_{ij}\} \) and \( D = \{\varepsilon_{ij}\} \) are the stress and strain tensors, respectively. Suppose

\[ \sigma_{ik} = a_k(x; \varepsilon_{ij}), \text{ where } \varepsilon_{ij} = \frac{u_j}{x_1} + \frac{u_i}{x_j}, \]

and \( u \) is the displacement vector.

Then the system corresponding to (1) has the form

(1') \[ D_ia_i(x; \varepsilon_{ij}) - f = 0, \]

where \( f \) is a vector of mass forces.

Denote

\[ T^2 = \inf_{\varepsilon > 0} \sup_{x, p, q} \sum_{i, k=1}^{m} \left| (p_i^1 + p_k^1) - (q_i^1 + q_k^1) - \varepsilon [a_k^1(x; p_j^1 + p_j^1) - a_k^1(x; q_j^1 + q_j^1)] \right|^2. \]

**THEOREM 2.** If

\[ 2T \left[ 1 + \left( \frac{m-2}{m-1} \right)^2 \right]^{1/2} \left[ 1 + \frac{1}{2} \left( 1 + \frac{m-2}{m+1} \right) \right]^{1/2} < 1, \]

then the displacement \( u \) (the weak solution of the problem (1')-(2)) is locally H"older-continuous on \( \Omega \).

As we have seen above, the conditions (5) and (8) are sharp. Nevertheless, there are the cases when the solution \( u \) has some additional properties. Suppose that \( \Gamma_s (s=1, \ldots, m-2) \) is a smooth \( s \)-dimensional manifold in \( \Omega \).

**THEOREM 3.** Let the conditions (4) be satisfied and let

\[ \frac{\Lambda - \lambda}{\Lambda + \lambda} \left[ 1 + \left( \frac{m-2-s}{m-1-s} \right)^2 \right]^{1/2} < 1. \]

Then \( u \in L_1(\Gamma_s) \) for all \( C^0_0 \) and \( \|u \|_{L_1(\Gamma_s)} \) is H"older continuous with respect to the coordinates orthogonal to \( \Gamma_s \).
Consider now the parabolic system

$$P(u) = \dot{u} - L(u) = 0$$

in the cylinder $Q = \Omega \times (0, T)$ with the initial condition $u|_{t=0} = 0$ and the condition (2). Suppose that the conditions (4) are satisfied.

**THEOREM 4.** There exists such a constant $C(m) > 0$ that if the inequality

$$\frac{\Lambda - \lambda}{\Lambda + \lambda} C(m) < 1$$

holds, then the weak solution $u$ of the problem

$$P(u) = 0, \quad u|_{t=0} = 0, \quad u|_{\partial \Omega} = g(x)$$

is for almost all $t \in [0, T]$ locally Hölder-continuous with respect to the variable $x$ in $\Omega$.

This theorem has some corollaries concerning the systems with hysteresis coefficients.