DIRICHLET'S PROBLEM ON A SNOWFLAKE

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1. If Ω is a domain in $\mathbb{R}^n$ with boundary Γ which is sufficiently smooth Dirichlet's problem in Ω in variational form was solved long ago (see for instance [4], [6], [7]). Recently J. Marschall [5] was able to treat the case when Ω is a Lipschitz domain. In this note we treat the case when Ω is von Koch's snowflake domain in $\mathbb{R}^2$. However, our method works for more general domains in $\mathbb{R}^n$ with fractal boundary. We refer to [8] for this fact and for proofs and further details on the material in this note. The results in [8] are based on extension and restriction theorems in [3] and [2].

2. Let Ω be any bounded open subset in $\mathbb{R}^n$ and consider the Dirichlet problem

$$\text{(1)} \qquad \begin{cases} \Delta u = -f & \text{in } \Omega \\ u = g & \text{on } \Gamma, \end{cases}$$

where f and g are given functions. In integrated form the first equation in (1) becomes after a partial integration

$$\int_\Omega \nabla u \cdot \nabla v dx = \int_\Omega f v dx, \quad \text{for} \quad v \in C_0^1(\Omega).$$

Together with the boundary condition $u=g$ on Γ this gives Dirichlet's problem in variational form.

3. Let $W_1^2(\Omega)$ be the Sobolev space with the usual norm of functions in $L^2(\Omega)$ having first order derivatives in $L^2(\Omega)$, and let $\mathring{W}_1^2(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in this norm.

Now we assume that $f \in L^2(\Omega)$ and that g defined on Γ can be extended to a function $g_E \in W_1^2(\Omega)$ in the sense that the trace (defined by (3) below) to Γ of g_E is g. Then, in the usual way it is proved by Hilbert space methods that there exists a unique $u \in W_1^2(\Omega)$ such that

$$\text{(2)} \qquad \begin{cases} \displaystyle\int_\Omega \nabla u \cdot \nabla v dx = \int_\Omega f v dx, & \text{for} \quad v \in \mathring{W}_1^2(\Omega), \quad \text{and} \\ u - g_E \in \mathring{W}_1^2(\Omega). \end{cases}$$

We want to answer the following questions when Γ is a fractal. How do we define the trace to Γ of functions in $W_1^2(\Omega)$ and what is the trace space to Γ of $W_1^2(\Omega)$? Does the solution $u \in W_1^2(\Omega)$ of (2) have trace g to Γ and does u depend uniquely on f and g?

4. From now on we assume that Ω is von Koch's snowflake domain in $\mathbb{R}^2$. This is the domain inside von Koch's curve Γ. To construct Γ we start from the boundary Γ_0 of an equilateral triangle with side of length 1. In the first step we get Γ_1 from Γ_0 by dividing each side of Γ_0 into three equal parts and replacing the middle part by the two other sides of an equilateral triangle having the middle part as base and the opposite corner outside Γ_0. This gives Γ_1 which consists of $3 \cdot 4$ sides of length 3^{-1}. In the second step we construct Γ_2 from Γ_1 in an analogous way, and so on. Γ_n consists of $3 \cdot 4^n$ sides of length 3^{-n} and the limit of Γ_n is Γ which is a fractal of Hausdorff dimension $d=\log 4/\log 3$ (see [1], p.118 for a picture, and [1] or [3] for the definition of Hausdorff measure and dimension).

5. We now define the trace to Γ of a function $u \in W_1^2(\Omega)$. We say that u can be strictly defined at $x \in \Omega \cup \Gamma$ if $B(x,r)$ denotes the closed disk with center x and radius r and the limit

$$(3) \qquad \tilde{u}(x) := \lim_{r \to 0} \frac{1}{m(B(x,r) \cap \Omega)} \int_{B(x,r) \cap \Omega} u(y)\,dy$$

exists. The <u>trace</u> $\mathrm{Tr}\, u$ of u to Γ is defined as the function $u|\Gamma$ given by

$$(u|\Gamma)(x) := \tilde{u}(x)$$

at every $x \in \Gamma$ where $\tilde{u}(x)$ exists. It may be proved ([8], Proposition 2.3) that $u|\Gamma$ is defined d-a.e. on Γ, i.e. everywhere on Γ except on a subset of d-dimensional Hausdorff measure zero.

6. Next we define the <u>Besov space</u> $B_\beta^{2,2}(\Gamma)$ where, for the rest of this note, we put

$$\beta = 1-(2-d)/2, \quad d=\log 4/\log 3.$$

We let m_d denote the d-dimensional Hausdorff measure and introduce the measure μ on Γ by

$$\mu(E) = m_d(E \cap \Gamma), \quad \text{for all Borel sets } E.$$

A function g defined d-a.e. on Γ belongs to $B^{2,2}_{\beta}(\Gamma)$ if it has finite norm

$$\|g;B^{2,2}_{\beta}(\Gamma)\| := \|g\|_{L^2(\mu)} + \left\{ \iint\limits_{|x-y|<1} \frac{|g(x)-g(y)|^2}{|x-y|^{d+2\beta}}\, d\mu(x)d\mu(y) \right\}^{1/2} .$$

7. The first part of the following basic theorem (see [8], Theorem 2.3 and 3.1) gives the trace space which we asked for in Section 3.

THEOREM 1. *The trace operator* $Tr: u \mapsto u|\Gamma$ *is a bounded linear surjection*

$$Tr: W^2_1(\Omega) \to B^{2,2}_{\beta}(\Gamma)$$

with a bounded linear right inverse. Furthermore, the kernel of the trace operator is $\mathring{W}^2_1(\Omega)$.

8. We now return to the Dirichlet problem in variational form. The connection in Section 3 between g_E and g is $g_E|\Gamma=g$ where $g_E|\Gamma$ is the trace of g_E to Γ in the sense given by (3). The trace space to Γ of $W^2_1(\Omega)$ is $B^{2,2}_{\beta}(\Gamma)$, $\beta=1-(2-d)/2$, by Theorem 1.

The condition $u-g_E \in \mathring{W}^2_1(\Omega)$ in (2) means, by the last part of Theorem 1, that the trace to Γ of $u-g_E$ is 0 which gives $u|\Gamma=g_E|\Gamma=g$, i.e. the trace to Γ of the solution $u \in W^2_1(\Omega)$ of (2) is g. From Theorem 1 it also follows in a standard way that the solution $u \in W^2_1(\Omega)$ of (2) depends uniquely on f and g.

Summing up and using Section 3 we get the following solution of Dirichlet's problem in variational form.

THEOREM 2. *Let* Ω *be von Koch's snowflake domain in* $\mathbb{R}^2$ *with boundary* Γ *with dimension* $d=\log 4/\log 3$. *Given* $f \in L^2(\Omega)$ *and* $g \in B^{2,2}_{\beta}(\Gamma)$, $\beta=1-(2-d)/2$, *the problem*

$$\begin{cases} \displaystyle\int_{\Omega} \nabla u \cdot \nabla v\,dx = \int_{\Omega} fv\,dx, & \text{for all } v \in \mathring{W}^2_1(\Omega) \\ u|\Gamma = g \end{cases}$$

has a unique solution $u \in W^2_1(\Omega)$.

It may also be proved that the mapping $\{f,g\} \mapsto u$ is a bounded linear mapping from $L^2(\Omega) \times B^{2,2}_{\beta}(\Gamma)$ to $W^2_1(\Omega)$.

References

[1] K.J. FALCONER, The Geometry of Fractal Sets, Cambridge Univ. Press, Cambridge, 1985.

[2] P.W. JONES, Quasiconformal mappings and extendability of functions in Sobolev spaces, Acta Math. 147, 1981, 71-88.

[3] A. JONSSON and H. WALLIN, Function Spaces on Subsets of $\mathbb{R}^n$, Math. Reports 2, Part 1, Harwood Acad. Publ., London, 1984.

[4] A. KUFNER, O. JOHN and S. FUČIK, Function Spaces, Noordhoff International Publishing, Leyden, 1977.

[5] J. MARSCHALL, The trace of Sobolev-Slobodeckij spaces on Lipschitz domains, Manuscripta Math. 58, 1987, 47-65.

[6] J. NEČAS, Les méthodes directes en théorie des equations elliptiques, Academia, Prague, 1967.

[7] P.A. RAVIART and J.M. THOMAS, Introduction à l'analyse numérique des équations aux dérivées partielles, Masson, Paris, 1983.

[8] H. WALLIN, The trace to the boundary of Sobolev spaces on a snowflake, Department of Mathematics, Univ. of Umeå, No. 5, 1989.