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ON THE MATHEMATICAL AND NUMERICAL MODELLING OF ELECTRON BEAMS

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1. Introduction

Electron guns, or more generally charged particle injectors, are of crucial importance in the technology of particle accelerators, free electron lasers, microwave tubes, ... They are required to produce relativistic electron beams of very high quality : high current and low emittance. Numerical simulations are now currently used in the design of electron guns : the most sophisticated mathematical models to solve are based on the time-dependent relativistic Vlasov-Maxwell system of equations in complex three-dimensional geometries; If we denote by $\Omega \subset \mathbb{R}^3$ the geometric domain under consideration, by Γ its boundary and $\mathbf{n}$ the unit outward normal to Γ, the problem consists in finding functions $f=f(x,p,t)$, $\mathbf{E}=\mathbf{E}(x,t)$, $\mathbf{B}=\mathbf{B}(x,t)$ solutions of

$$(1.1) \qquad \frac{\partial f}{\partial t} + \mathbf{v}.\nabla_x f + \mathbf{F}.\nabla_p f = 0, \quad x \in \Omega,\ \mathbf{p} \in \mathbb{R}^3,\ t > 0$$

and

$$(1.2) \qquad c^{-2}\frac{\partial \mathbf{E}}{\partial t} - \nabla \times \mathbf{B} = -\mu_0 \mathbf{J}$$

$$(1.3) \qquad \frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0$$

$$(1.4) \qquad \nabla . \mathbf{E} = \varepsilon_0^{-1}\rho$$

$$(1.5) \qquad \nabla . \mathbf{B} = 0, \quad x \in \Omega, \quad t > 0.$$

In (1.1)-(1.5), f is the electron distribution function, **E** the electric field, **B** the magnetic field and

$$(1.6) \qquad \mathbf{v} = c(p^2 + m^2c^2)^{-1/2}\,\mathbf{p}, \quad p = |\mathbf{p}|$$

$$(1.7) \qquad \mathbf{F} = -e(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

$$(1.8) \qquad \rho = -e\int f dp, \qquad \mathbf{J} = -e\int \mathbf{v} f dp.$$

Concerning the boundary conditions, we assume that on a part Γ_0 of Γ (the cathode) electron emission occurs while on $\Gamma_1 = \Gamma - \Gamma_0$ the electrons are free to leave the domain, i.e., we prescribe

$$(1.9) \qquad f(x,p,t) = \begin{cases} g(x,p,t), & \mathbf{p}.\mathbf{n} < 0, \ x \in \Gamma_0 \\ 0, & \mathbf{p}.\mathbf{n} < 0, \ x \in \Gamma_1 \end{cases}$$

for some given emission function g. On the other hand, we assume for simplicity that Γ is a perfect conductor so that

$$(1.10) \qquad \mathbf{E} \times \mathbf{n} = 0 \quad \text{on } \Gamma, \quad t > 0.$$

Adding to the equations (1.1)-(1.10) natural initial conditions leads to a problem whose solution presents interesting difficulties of both mathematical and numerical nature. We describe here some new results in these two directions.

2. The plane diode

The mathematical analysis of the above system of equations is in fact far from being well understood. Hence we restrict ourselves to the simplest possible model : the stationary Vlasov-Poisson equations for a plane diode

(2.1) $$v\frac{\partial f}{\partial x}+\frac{e}{m}\frac{d\phi}{dx}\frac{\partial f}{\partial v}=0, \quad x\in(0,L), \quad v\in\mathbb{R}$$

(2.2) $$f(0,v)=g(v), \quad v>0, \quad f(L,v)=0, \quad v<0$$

(2.3) $$\frac{d^2\phi}{dx^2}=\frac{1}{\varepsilon_0}ne, \quad n=\int f dv, \quad x\in(0,L)$$

(2.4) $$\phi(0)=0, \quad \phi(L)=\phi_L.$$

Here the cathode is located at the point $x = 0$ and the magnetic effects have been neglected so that $E=-\frac{d\phi}{dx}$ where ϕ is the electric potential. One can prove (cf. [4]).

Theorem 1. The following properties hold :

(i) *Assume that the function* g *satisfies for all* $v > 0$

(2.5) $$0\le g(v)\le c(1+v)^{-\lambda}, \quad \lambda>1.$$

Then problem (2.1)-(2.4) *has at least one solution.*

(ii) *Uniqueness holds in the class of positive potential solutions.*

(iii) *If* g *is decreasing on* $]0,\infty[$, *the solution is unique.*

In practice, we have

$$\varepsilon^2=(2e\phi_L)^{-1}mv_{th}^2 \ll 1, \quad v_{th}^2=\left(\int_0^\infty g(v)dv\right)^{-1}\int_0^\infty v^2 g(v)dv$$

and there exists in the neighborhood of the cathode a boundary layer of width $0(\varepsilon^{3/2})$ in which a part of the emitted electrons return to the cathode. A proper scaling of the Vlasov-Poisson equations leads to the singular perbutation problem :

$$v\frac{\partial f^\varepsilon}{\partial x}+\frac{1}{2}\frac{d\phi^\varepsilon}{dx}\frac{\partial f^\varepsilon}{\partial v}=0, \quad x\in(0,1), \quad v\in\mathbb{R}$$

$$f^\varepsilon(0,v)=\varepsilon^{-2}g(\varepsilon^{-1}v), \quad v>0, \quad f^\varepsilon(1,v)=0, \quad v<0$$

$$\frac{d^2\phi^2}{dx^2} = n^\varepsilon, \quad n^\varepsilon = \int f^\varepsilon dv, \quad x \in (0,1)$$

$$\phi^\varepsilon(0) = 0, \quad \phi^\varepsilon(1) = 1.$$

Setting $i_g = \int vg(v)dv$, we notice that $g^\varepsilon(v) = \varepsilon^{-2}g(\varepsilon^{-1}v)$ converges formally to $i_g v^{-1}\delta(v)$ which is not a well defined distribution. Concerning the convergence of $(f^\varepsilon,\phi^\varepsilon)$ towards the solution of a reduced problem, we have (cf. [3]).

***Theorem 2**. Assume that* g *is a decreasing function. Then, as* $\varepsilon \to 0$, $f^\varepsilon \to f^0$ *in* $\mathcal{M}_b([0,1] \times \mathbb{R})$ *weak-star,* $\phi^\varepsilon \to \phi^0$ *in* $C^0([0,1])$ *strongly where* (f^0,ϕ^0) *is a distributional solution of the Vlasov-Poisson equations which satisfies*

$$f^0(x,v) = i^0(\phi^0(x))^{-1/2}\, \delta(v-\phi(x)^{1/2}), \quad i^0 = \min(i_g,\tfrac{4}{9})$$

$$\frac{d^2\phi^0}{dx^2} = i^0\, \phi_0(x)^{-1/2}, \quad \phi^0(0) = 0, \quad \phi^0(1) = 1.$$

3. A coupled particle-finite element method

For solving numerically the Vlasov-Maxwell equations, we propose a coupled particle-finite element method. We approximate the electron distribution function f by a linear combination of Dirac measures in the phase-space

(3.1) $$f^h(x,p,t) = \sum_k \alpha_k\, \delta(x-x_k(t))\, \delta(p-p_k(t)),$$

where

(3.2) $$\frac{dx_k}{dt} = v_k, \quad \frac{dp_k}{dt} = F(x_k,p_k,t).$$

Hence, ρ and J are approximated by

(3.3) $$\rho^h(x,t) = -e \sum_k \alpha_k\, \delta(x-x_k(t)), \quad J^h(x,t) = -e \sum_k \alpha_k v_k\, \delta(x-x_k(t)).$$

On the other hand, Maxwell equations are solved by means of a finite element method using an unstructured mesh. In fact, the equations (1.2)-(1.4) lead to a well-posed problem if and only if the following charge conservation equation holds

(3.4) $$\frac{\partial \rho}{\partial t} + \nabla . J = 0.$$

Unfortunately, the computed charge and current densities do not satisfy the equation (3.4) at least in general. Hence, we need to introduce a pseudo-displacement current $-\nabla\phi$ in (1.2) which becomes

(3.5) $$c^{-2}\left(\frac{\partial E}{\partial t} - \nabla\phi\right) - \nabla \times \mathbf{B} = -\mu_0 \mathbf{J}$$

together with the boundary condition

(3.6) $$\phi = 0 \quad \text{on} \quad \Gamma, t > 0.$$

__*Theorem 3*__*. Assume that* ρ *and* $\mathbf{J}$ *are given. Then, the problem* (1.3), (1.4), (1.5), (3.4) *with the boundary conditions* (1.10), (3.5) *and natural initial conditions is well posed.*

One can now solve this new problem by using a standard finite element method. For instance, assuming for simplicity that the problem is two-dimensional, we approximate $B_z = B_z(x,y)$; $\phi = \phi(x,y)$ by continuous piecewise linear functions and $\mathbf{E} = (E_x, E_y)$ by piecewise constant functions on a triangular mesh.

It remains to couple the particle method and the finite element method by defining continuous representations of ρ^h and $\mathbf{J}^h$ from their particle representations (3.3) and also to derive a particle discretization of the boundary condition (1.9). For a thorough discussion of the method in the axisymmetric case and various numerical tests, we refer to [1]. See also [2] for a review on the numerical simulation of kinetic equations by particle methods.

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