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Outer Measure on Boolean Algebras

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We show that it is consistent that there is an atomless outer measure on a complete, ccc, (ω, ∞)-weakly distributive, countably generated, atomless Boolean algebra without any Maharam submeasure.

1. Introduction

A problem of the existence of Maharam submeasure on a Boolean algebra is a classical problem of the set-theoretic measure theory. This subject was started by D. Maharam.

In [M], D. Maharam pointed out that there exists a strictly positive Maharam submeasure on a complete Boolean algebra B, exactly in the case when the order sequential topology τ_s (in short os-topology) on B is metrizable. She asked whether the existence of an atomless outer measure on a complete, ccc, (ω, ∞)-weakly distributive, atomless Boolean algebra B imply the existence of a Maharam submeasure on algebra B ? - cf. [M], sec. 8.1.

By Balcar, Jech and Pazák [B-J-P], it is consistent that every complete, ccc, (ω, ∞)-weakly distributive Boolean algebra B carries a Maharam submeasure on B. So the affirmative answer on the Maharam question is consistent. We show here that this question is in fact independent of ZFC.

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2. Basic facts

In this paper we use the same notions as in [B-G-J] and [B-J-P]. For the reader convenience we repeat some facts and notions here. For other notions and basic facts of Boolean algebras theory see [Ko] or [V].

Let B be a Boolean algebra. A *submeasure* on B is a function $\mu : B \to \mathbf{R}^+$ with the following properties:

(i) $\mu(b) = 0$ if and only if $b = \mathbf{0}$,

(ii) $\mu(a) \leq \mu(b)$ whenever $a \leq b$,

(iii) $\mu(a \vee b) \leq \mu(a) + \mu(b)$.

A submeasure μ on a σ-complete Boolean algebra B is

(iv) *outer measure* if $\lim \mu(a_n) = \mu(b)$ for every increasing sequence $\{a_n : n \in \omega\}$, such that $\bigvee \{a_n : n \in \omega\} = b$,

(v) *Maharam submeasure* or *continuous* if $\lim \mu(a_n) = 0$ for every decreasing sequence $\{a_n : n \in \omega\}$, such that $\bigwedge \{a_n : n \in \omega\} = \mathbf{0}$.

Every σ-additive strictly positive measure on a σ-complete Boolean algebra is a Maharam submeasure and every Maharam submeasure is an outer measure, but not conversely.

Let B be a complete Boolean algebra. Then B is (ω,κ)-weakly distributive if it satisfies the following distributive law for cardinal κ,

$$\bigwedge_n \bigvee_\alpha a_{n\alpha} = \bigvee_{f:\omega\to[\kappa]^{<\omega}} \bigwedge_n \bigvee_{\alpha\in f(n)} a_{n\alpha}.$$

A complete Boolean algebra is (ω,∞)-weakly distributive if it is (ω,κ)-weakly distributive for every cardinal κ.

Every σ-complete Boolean algebra B which carries a Maharam submeasure is ccc and (ω,∞)-weakly distributive. See [Fr1] for details. But σ-complete Boolean algebra with an outer measure may be neither (ω,∞)-weakly distributive nor ccc.

We also use some topological notions. The notation is based on the Engelking's book [E], Vladimirov's [V] and a paper [B-G-J]. Let us repeat some basics:

Let (X,τ) be a topological space. The space (X,τ) is a *Fréchet* space if for every $A \subseteq X$ an element $x \in \mathrm{cl}_\tau(A)$ iff $x_n \xrightarrow[\tau]{} x$ for some sequence $\{x_n\}_{n\in\omega}$, of elements of a set A.

We say that a sequence $\{b_n\}_{n\in\omega}$ of elements of σ-complete Boolean algebra B algebraically converges to an element $b \in B$ if and only if

$$b = \bigwedge_{k\in\omega} \bigvee_{n\geq k} b_n = \bigvee_{k\in\omega} \bigwedge_{n\geq k} b_n.$$

We write then $b_n \Rightarrow b$.

The order sequential topology (in short os-topology) τ_s is the largest topology τ on B such that if the sequence $\{b_n\}$ converges to b algebraically then it also converges to b in the topology; i.e. if $b_n \Rightarrow b$ then $b_n \xrightarrow[\tau]{} b$. The topology τ_s is T_1, i.e. every singleton is a closed set.

Usually it is not true that the convergence '$\xrightarrow[\tau_s]{}$' in the topology τ_s implies the algebraic convergence '$\Rightarrow$'. It is well known that a sequence $\{x_n\}$ converges to x topologically if and only if every subsequence of $\{x_n\}$ has a subsequence that converges to x algebraically.

Below we give some simple facts about the algebraic convergence (see [M]):

(i) $b_n \Rightarrow \mathbf{0}$ iff $\bigwedge\limits_{k\in\omega} \bigvee\limits_{n\geq k} b_n = \mathbf{0}$;

(ii) if the b_n's are pairwise disjoint then $b_n \Rightarrow \mathbf{0}$;

(iii) if $b_n \Rightarrow b$ and $a_n \Rightarrow a$ then $b_n \vee a_n \Rightarrow b \vee a$ and $-b_n \Rightarrow -b$;

(iv) if $\{b_n\}$ is increasing then $b_n \Rightarrow \bigvee\limits_{n\in\omega} b_n$.

For any subset A of the algebra B let

$$u(A) = \{x : x \text{ is the limit of a sequence } \{x_n\} \text{ of elements of } A\}.$$

The closure of a set A in the topology τ_s is obtained by an iteration of u:

$$cl_{\tau_s}(A) = \bigcup_{\alpha<\omega_1} u^{(\alpha)}(A),$$

where $u^{(\alpha+1)}(A) = u(u^{(\alpha)}(A))$, and for a limit α, $u^{(\alpha)}(A) = \bigcup_{\beta<\alpha} u^{(\beta)}(A)$. Moreover, the topological space (B, τ_s) is Fréchet if and only if $cl(A) = u(A)$ for every $A \subseteq B$.

For any subalgebra A of a complete, ccc Boolean algebra B, the closure $cl_{\tau_s}(A)$ of A in the topology τ_s is a subalgebra completely generated by A. Clearly it is $u(A)$ if (B, τ_s) is a Fréchet space.

For every Maharam submeasure $\mu : B \to \mathbf{R}^+$ the following function $d_\mu : B \times B \to \mathbf{R}^+$ given by formula: $d_\mu(a,b) = \mu(a \triangle b)$, for any $a, b \in B$, is a metric on B. The topology given by d_μ coincides with the order sequential topology (see [V]; sec. 4.2.5 and 7.1.1). Hence if there exists any Maharam submeasure on B, then (B, τ_s) is metrizable. Moreover, any Maharam submeasures μ_1, μ_2 on B give the same topology τ_s on B.

It is proved in [B-G-J] that for a complete, ccc Boolean algebra B, if the space (B, τ_s) is T_2, then it is a metrizable space.

3. Outer measure

Let $B^+ = B - \{\mathbf{0}\}$. Every Boolean algebra B carries a submeasure $\mu : B \to \mathbf{R}^+$ defined by the formula $\mu(b) = 1$ for every $x \in B^+$ and $\mu(\mathbf{0}) = 0$.

Definition 3.1. *Let B be an atomless Boolean algebra. A submeasure $\mu : B \to \mathbf{R}^+$ is called atomless if for every $b \in B^+$ there is an element $a \in B^+$ such that $a < b$ and $\mu(a) < \mu(b)$*

Let the abbreviation SH means the Souslin Hypothesis i.e. the statement that there is no Souslin algebra (see [J]).

Maharam proved in [M] (see also ch.8 sec. 1.4, pp. 396 – 398 in [V]) the following:

Theorem 3.2. *If every complete, atomless, (ω, ∞)-weakly distributive, ccc Boolean algebra B carries an atomless submeasure then SH is true.*

Maharam in the proof of the above theorem established that if SH is not true, then there is no atomless submeasure on the Souslin algebra. Where the Souslin algebra is a complete, atomless, ccc, ω – distributive Boolean algebra.

On the other hand, Balcar, Jech, Pazák in [B-J-P] showed:

Theorem 3.3. *Con (ZFC + every complete, (ω, ∞)-weakly distributive, ccc Boolean algebra B carries a Maharam submeasure).*

Note that a Maharam submeasure on an atomless Boolean algebra is atomless.

Below, we show the relative consistency of a complete, atomless, (ω, ∞)-weakly distributive, ccc Boolean algebra B with an atomless submeasure but without any Maharam submeasure.

Let $\mathbf{b}$ be the *bounding number.* We repeat from [B-G-J]:

Theorem 3.4. *Let B be a complete Boolean algebra. The space (B, τ_s) in the sequential order topology is Fréchet if and only if the algebra B is (ω, ω)-weakly distributive and satisfies the $\mathbf{b}$-chain condition.*

As a consequence of the fact that a complete, ccc, (ω, ∞)-weakly distributive Boolean algebra is a Fréchet space in os-topology, we obtained the following lemma several years ago. Now it is an element of mathematical folklore. A graceful proof of part i) of the next lemma can be found in the Pazák's Ph.D. thesis (see:[P] theorem 3.37, p. 23). The second part is a direct consequence of the fact that we assume that algebra B is atomless.

Lemma 3.5. *Let B be a complete, atomless, (ω, ∞)-weakly distributive, ccc Boolean algebra and let A be a subalgebra of B which completely generates B. Then*

(i) *The set $\{\bigwedge\{a_n : n \in \omega\} : \{a_n : n \in \omega\} \in [A]^{\omega}\}$ is a dense subset of B.*

(ii) *For every element $b \in B^+$ there is an element $a \in A^+$ such that $\mathbf{0} < b - -(a \wedge b) < b$.*

In fact for each element $b \in B^+$ we have infinitely many elements $a \in A^+$ in the subalgebra A which satisfy the above condition (ii).

Using lemma 3.5 we show:

Proposition 3.6. *Every complete, atomless, (ω, ∞)-weakly distributive, ccc, completely countably generated Boolean algebra B carries an atomless outer measure.*

Proof. For every countable set X, which completely generates the Boolean algebra B the subalgebra generated by the set X is a countable Boolean algebra, which completely generates the whole algebra B. Let $\{g_n : n \in \omega\}$ be a given enumeration of the subset $G - \{\mathbf{0}\}$of a countable subalgebra G of algebra B, which completely generates B. We recall that by theorem 3.4 the topological space (B, τ_s) is Fréchet and $B = cl_{\tau_s}(G) = u(G)$. Hence for every $b \in B$ there is a sequence $\{b_n : n \in \omega\}$ of elements of subalgebra G such that

$$b = \bigwedge_{k\in\omega} \bigvee_{n\geq k} b_n = \bigvee_{k\in\omega} \bigwedge_{n\geq k} b_n$$

Which follows from the lemma 3.5. part (i). Hence for every $b \in B^+$ the set $\{n\in \omega : g_n \wedge b \neq \mathbf{0}\}$ is infinite.

For every $n \in \omega$ define a function $\mu_n : B \to \{0,1\}$ as follow $\mu_n(b) = 1$ if $g_n \wedge b \neq \mathbf{0}$ and $\mu_n(b) = 0$ if $g_n \wedge b = \mathbf{0}$.

Now let a function $\mu : B \to \mathbf{R}^+$ be done by formula $\mu(b) = \sum_{n\in\omega} 2^{-(n+1)} \mu_n(b)$. The above function μ, satisfies the conditions of outer measure.

Because, by the lemma 3.5 ii) for every $b \in B^+$ there is $n \in \omega$ such that $0 < b - (g_n \wedge b) < b$ then for $a = b - (g_n \wedge b)$, $a < b$ and by the definition of μ, $\mu(a) < \mu(b)$. So the outer measure μ is an atomless submeasure. □

Let MA respectively $\neg CH$ abbreviate the Martin's Axiom respectively the negation of the continuum hypothesis.

Proposition 3.7. *Let $MA + \neg CH$ be true. For every σ-saturated σ-ideal I on the power algebra $P(X)$, where X is a subspace of the reals $\mathbf{R}$ of the cardinality κ, $\omega < \kappa < 2^\omega$, the quotient algebra $B = P(X)/I$ is a complete (ω, ∞)-weakly distributive, ccc, completely countably generated Boolean algebra.*

Proof. If $MA + \neg CH$ is true then $\mathbf{b} = 2^\omega$. For any subspace $X \subset \mathbf{R}$ of the cardinality κ, by theorem 3.4 the space $((P(X), \tau_s)$ is Fréchet. The set X is a $\mathbf{Q}$-set i.e. every subset of X is a G_δ subset of X. So the space $((P(X), \tau_s) = (\mathbf{B}(X), \tau_s)$ is a separable space. Topology τ_s in $B = P(X)/I$ is the quotient topology of the topology τ_s in $P(X)$ and the natural mapping is open. Hence (B, τ_s) is a separable Fréchet space and the quotient algebra B is a complete (ω, ∞)-weakly distributive, ccc, completely countably generated Boolean algebra. □

Let us note that the Martin's Axiom and the negation of the continuum hypothesis $MA + \neg CH$ imply the Souslin hypothesis SH.

Theorem 3.8. *If Con(ZFC + there exists a measurable cardinal) then Con(ZFC + SH + there exists a complete, atomless, (ω, ∞)-weakly distributive, ccc, completely countably generated Boolean algebra B which carries an atomless outer measure without any Maharam submeasure).*

Proof. Let M be a countable transitive model with measurable cardinal κ such that $2^\kappa = \kappa^+$. Let I be a nonprincipal κ-complete prime ideal over κ. By *ccc* forcing we can obtain a generic extension $M[G]$ for $MA + \kappa < 2^{\omega}$ (see [Ku]). Then in $M[G]$ the quotient algebra $P(\kappa)/J$, where ideal J is defined as $x \in J$ iff $x \subset y$ for some $y \in I$, is a complete, atomless, (ω, ∞)-weakly distributive, ccc, completely countably generated Boolean algebra B. By Proposition 3.6 $P(\kappa)/J$ carries an atomless outer measure. It is well known ([G]; see also [B-J], [Vel] or [Fr3]), that there is no Maharam submeasure on $P(\kappa)/J$. Because $MA + \neg CH$ holds in $M[G]$, then in $M[G]$ the Souslin hypothesis SH is true. $\square$

The theorems 3.3 and 3.8 show that the answer to Maharam's question: *Whether the existence of an atomless outer measure on a complete, ccc, (ω, ∞)-weakly distributive, atomless Boolean algebra B imply the existence of a Maharam submeasure on algebra B?* is independent of the axioms of ZFC.

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