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GUARANTEED AND FULLY COMPUTABLE TWO-SIDED BOUNDS OF FRIEDRICHS' CONSTANT

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Abstract

This contribution presents a general numerical method for computing lower and upper bound of the optimal constant in Friedrichs' inequality. The standard Rayleigh-Ritz method is used for the lower bound and the method of *a priori-a posteriori inequalities* is employed for the upper bound. Several numerical experiments show applicability and accuracy of this approach.

1. Introduction

From the numerical point of view, the guaranteed and fully computable two-sided bounds always provide a strong information about the computed quantity. Their difference is a reliable bound on the approximation error and in applications they allow to stay on the safe side by using properly either the lower or the upper bound as the approximation.

In this contribution, we concentrate on the optimal constant in Friedrichs' inequality. The presented two-sided bounds are guaranteed up to round-off errors. The chosen approach is quite general and theoretically it can be used in arbitrary dimension, for any domain, and for different variants of Friedrichs' inequality. Practically, we are limited by particular choices of discretization methods. For instance, the presented numerical examples are limited to polygonal domains in two dimensions.

The optimal constant in Friedrichs' inequality is called Friedrichs' constant and its value is connected with the smallest eigenvalue of the corresponding differential operator. The classical Rayleigh-Ritz method provides an upper bound on the exact eigenvalue and consequently a lower bound for Friedrichs' constant.

Computing a lower bound of the smallest eigenvalue and hence computing the upper bound of Friedrichs' constant is considerably more difficult task. We use the method of *a priori-a posteriori inequalities* [5, 9]. The original idea relies on C^2-smooth test and trial functions, which are technically difficult to work with. Therefore, we proposed in [11] an alternative approach based on complementarity and standard Raviart-Thomas finite element method.

We briefly review this approach in Sections 2–4 and provide several numerical experiments in Sections 5–7.

2. Friedrichs' inequality

Let us consider a domain $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary. Further, let Γ_{D} and Γ_{N} be two relatively open and disjoint subsets of the boundary $\partial\Omega$ such that $\partial\Omega = \overline{\Gamma}_{\mathrm{D}} \cup \overline{\Gamma}_{\mathrm{N}}$. Let the $(d-1)$-dimensional measure of Γ_{D} be positive. We will refer Γ_{D} and Γ_{N} to as Dirichlet and Neumann parts of the boundary, respectively. Further, we consider Sobolev space $H^1(\Omega) = \{v \in L^2(\Omega) : \nabla v \in [L^2(\Omega)]^d\}$ and its subspace $V = \{v \in H^1(\Omega) : v|_{\Gamma_{\mathrm{D}}} = 0\}$ of functions with vanishing traces on Γ_{D}.

In this contribution, we will assume the following variant of Friedrichs' inequality:

$$\|v\|_{0,\Omega} \le C_{\mathrm{F}} \|\nabla v\|_{0,\Omega} \quad \forall v \in V, \tag{1}$$

where $\|\cdot\|_{0,\Omega}$ stands for the $L^2(\Omega)$-norm. Let us note that this inequality is named after Kurt O. Friedrichs [3]. The optimal (smallest) possible value of the constant C_{F} such that inequality (1) holds is called Friedrichs' constant and the symbol C_{F} will denote this optimal value throughout the paper. The particular value of C_{F} depends on the domain Ω and on the Dirichlet part of the boundary Γ_{D}.

Friedrichs' constant scales naturally with the size of Ω. Namely, if $\widetilde{\Omega} = k\Omega$, $\widetilde{\Gamma}_{\mathrm{D}} = k\Gamma_{\mathrm{D}}$, and $\widetilde{\Gamma}_{\mathrm{N}} = k\Gamma_{\mathrm{N}}$ for some $k \in \mathbb{R}$ then Friedrichs' constants $\widetilde{C}_{\mathrm{F}}$ and C_{F} corresponding to $\widetilde{\Omega}$ and Ω, respectively, satisfy $\widetilde{C}_{\mathrm{F}} = kC_{\mathrm{F}}$.

In special cases, Friedrichs' constant can be computed analytically. For example, it was computed for a rectangle, circle, and a circular wedge in [6] for $\Gamma_{\mathrm{D}} = \partial\Omega$. Result [8] can be used for analytic computation of C_{F} for equilateral and right-angle triangles. In certain simple cases (e.g. rectangle) it can be computed even if $\Gamma_{\mathrm{D}} \neq \partial\Omega$. In less special cases there are analytic upper bounds for Friedrichs' constant. The Faber-Kran inequality [2, 4] yields upper bound $C_{\mathrm{F}} \le \sqrt{|\Omega|}/(j_{0,1}\sqrt{2\pi})$, where $|\Omega|$ is the area of the two-dimensional domain Ω and $j_{0,1} \doteq 2.404826$ is the first positive root of the Bessel function J_0. Similarly, in [7] we can find an estimate $C_{\mathrm{F}} \le \pi^{-1}\left(|a_1|^{-2} + \cdots + |a_d|^{-2}\right)^{-1/2}$, where $|a_1|, \ldots, |a_d|$ are lengths of sides of a d-dimensional box in which the domain Ω is contained. Note that both these estimates require $\Gamma_{\mathrm{D}} = \partial\Omega$. However, in more general cases the value of Friedrichs' constant has to be computed numerically.

3. Lower bound on Friedrichs' constant

Friedrichs' constant C_{F} from (1) is connect with the smallest eigenvalue of the Laplace eigenvalue problem that can be formulated in a weak sense as: find $\lambda_i \in \mathbb{R}$ and $u_i \in V$, $u_i \neq 0$, $i = 1, 2, \ldots$, such that

$$(\nabla u_i, \nabla v) = \lambda_i (u_i, v) \quad \forall v \in V, \tag{2}$$

where the parenthesis denote the $L^2(\Omega)$ inner product. If $\lambda_1 = \min_i \lambda_i$ stands for the smallest eigenvalue of (2) then it can be easily shown, see e.g. [10, 11], that

$$C_{\mathrm{F}} = 1/\sqrt{\lambda_1}. \tag{3}$$

A standard method for computing approximations of the eigenvalues λ_i is the Rayleigh-Ritz method. In this method we consider a finite dimensional subspace $V^h \subset V$ and seek $\lambda_i^h \in \mathbb{R}$ and $u_i^h \in V^h$, $u_i^h \neq 0$ such that

$$(\nabla u_i^h, \nabla v^h) = \lambda_i^h (u_i^h, v^h) \quad \forall v^h \in V^h.$$

This is equivalent to the generalized eigenvalue problem $Ax_i = \lambda_i^h M x_i$ for the stiffness and mass matrices A and M. If a standard finite element method is used then matrices A and M are sparse and efficient methods of numerical linear algebra can be used. The Rayleigh-Ritz method is well known for providing an upper bound on the smallest eigenvalue. Indeed, since the differential operator in (2) and the corresponding matrices A and M are symmetric, we can express λ_1 and λ_1^h as minima of (generalized) Rayleigh quotients over V and V^h, respectively, and we obtain

$$\lambda_1 = \min_{\substack{v \in V \\ v \neq 0}} \frac{(\nabla v, \nabla v)}{(v, v)} \leq \min_{\substack{v^h \in V^h \\ v^h \neq 0}} \frac{(\nabla v^h, \nabla v^h)}{(v^h, v^h)} = \min_{\substack{x \in \mathbb{R}^n \\ x \neq 0}} \frac{x^T A x}{x^T M x} = \lambda_1^h,$$

where $n = \dim V^h$. Consequently, the approximation $C_{\mathrm{F}}^{\mathrm{low}} = (\lambda_1^h)^{-1/2}$ of Friedrichs' constant, see (3), is a lower bound on the exact value C_{F}, i.e.,

$$C_{\mathrm{F}}^{\mathrm{low}} = (\lambda_1^h)^{-1/2} \leq C_{\mathrm{F}}.$$

4. Upper bound on Friedrichs' constant

Computing an upper bound of Friedrichs' constant is a more difficult task, because it corresponds to the computation of a lower bound of the smallest eigenvalue. We employ the method of *a priori-a posteriori inequalities* [5, 9] enhanced by the complementary approach. Mathematical details, relations, and derivations can be found in [11]. Here, we just briefly describe the algorithm.

First, use the Rayleigh-Ritz method and compute approximations $\lambda_1^h \in \mathbb{R}$ and $u_1^h \in V$ of the smallest eigenvalue λ_1 and the corresponding eigenfunction u_1. Second, choose a flux reconstruction $\boldsymbol{q}_h \in \mathbf{H}(\mathrm{div}, \Omega) = \{\boldsymbol{q} \in [L^2(\Omega)]^d : \mathrm{div}\, \boldsymbol{q} \in L^2(\Omega)\}$. Third, compute

$$\alpha = \frac{\|\nabla u_1^h - \boldsymbol{q}_h\|_{0,\Omega}}{\|u_1^h\|_{0,\Omega}}, \quad \beta = \frac{\|\lambda_1^h u_1^h + \mathrm{div}\, \boldsymbol{q}_h\|_{0,\Omega}}{\|u_1^h\|_{0,\Omega}}, \quad X_2 = \frac{1}{2}\sqrt{\alpha^2 + 4(\lambda_1^h - \beta)} - \frac{\alpha}{2}.$$

The lower bound on the smallest eigenvalue and the corresponding upper bound on Friedrichs' constant are then given as

$$X_2^2 \leq \lambda_1 \quad \text{and} \quad C_{\mathrm{F}} \leq C_{\mathrm{F}}^{\mathrm{up}} = 1/X_2.$$

Although any $\boldsymbol{q}_h \in \mathbf{H}(\mathrm{div}, \Omega)$ provides an upper bound on C_{F}, an accurate approximation is obtained for an appropriate choice of $\boldsymbol{q}_h$, only. In this contribution, we consider a Raviart-Thomas finite element subspace $W_h \subset \mathbf{H}(\mathrm{div}, \Omega)$ based on

a triangulation of Ω and minimize $\alpha^2 + \beta^2$ over W_h. This minimization is equivalent to finding $\boldsymbol{q}_h \in W_h$ such that

$$(\operatorname{div} \boldsymbol{q}_h, \operatorname{div} \boldsymbol{\psi}_h) + \lambda_1^h(\boldsymbol{q}_h, \boldsymbol{\psi}_h) = \lambda_1^h(\nabla u_1^h, \boldsymbol{\psi}_h) - \lambda_1^h(u_1^h, \operatorname{div} \boldsymbol{\psi}_h) \quad \forall \boldsymbol{\psi}_h \in W_h.$$

This problem can be solved by standard finite element technology, see e.g. [1]. We note that this particular flux reconstruction is a brute force solution and if the efficiency is an issue then a local reconstruction based on ∇u_1^h has to be used.

Further, it is important to note that the method of a priori-a posteriori inequalities is justified only if the approximation λ_1^h is sufficiently accurate. In particular, the closest eigenvalue to λ_1^h must be λ_1. If λ_1 and the second smallest eigenvalues λ_2 are well separated then sufficiently accurate Rayleigh-Ritz approximations of λ_1 and λ_2 can provide good confidence about the validity of this assumption. In all numerical experiments present below we experienced exactly this situation.

5. Example A: Friedrichs' constant for triangles

Friedrichs' constant C_{F} depends on the size and shape of the domain Ω and on the size, shape, and position of Γ_{D}. The dependence on the size of Ω is well known, see Section 2. Therefore, the following numerical experiments concentrate on the dependence of C_{F} on the shape of Ω (Examples A and B) and on Γ_{D} (Example C).

In all experiments below, the Rayleigh-Ritz approximations λ_1^h and u_1^h are computed by linear finite elements on triangular meshes and the reconstructed fluxes $\boldsymbol{q}_h$ by quadratic Raviart-Thomas finite elements on the same triangular mesh.

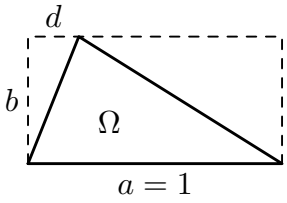

Figure 1: The shape of triangle Ω is given by the parameters b and d.

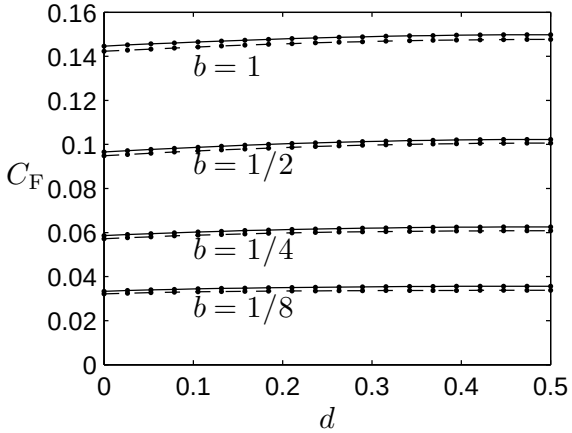

Figure 2: Friedrichs' constant for triangles with vertices $[0,0]$, $[1,0]$, $[d,b]$. Solid and dashed lines correspond to upper and lower bounds of C_{F}, respectively.

First, we consider Ω to be a nonobtuse triangle and assume $\Gamma_{\mathrm{D}} = \partial\Omega$. We investigate the dependence of C_{F} on the shape of this triangle. In particular, we consider triangles inscribed into a rectangle with lengths of sides a and b. The triangles have vertices with coordinates $(0,0)$, $(a,0)$, (d,b), see Figure 1. In particular, we fix $a = 1$,

consider four values of $b \in (0, 1]$, namely $b = 1, 1/2, 1/4, 1/8$, and 20 equidistributed values of d in $[0, 1/2]$. The two-sided bounds on C_{F} for the resulting triangles are presented in Figure 2. These bounds were computed on uniform meshes obtained by six successive uniform refinement steps of the original triangle. Thus, all these meshes have $4^6 = 4096$ triangles. We see that a fixed value of b yields triangles with the same area and the parameter d then controls the shape only. However, the observed dependence of C_{F} on the shape is negligible. We see a considerable dependence of C_{F} on b, but it is connected with the size of Ω as mentioned in Section 2.

6. Example B: Friedrichs' constant for regular stars

The value of Friedrichs' constant is of interest especially for nonconvex domains. Therefore, we consider Ω to be n-fold regular star with $n = 3, 4, \ldots, 30$ and choose $\Gamma_{\mathrm{D}} = \partial\Omega$. We put the outer vertices of stars Ω on a circle with radius $r_{\mathrm{out}} = 1$ and the inner vertices on a circle with radius $r_{\mathrm{in}} = 1/3$, see Figure 3. We use uniform mesh with $4^6 \cdot 2n$ triangles and compute both lower and upper bound on C_{F}.

Figure 4 shows the dependence of C_{F} on n. The value of C_{F} decreases with n and it seems that in the limit $n \to \infty$ it converges to Friedrichs' constant of a circle with radius $r_{\mathrm{in}} = 1/3$, which is approximately 0.138610. We note that Friedrichs' constant for a circle with radius $r_{\mathrm{out}} = 1$ is approximately 0.415831. The increasing gap between the lower and upper bound of C_{F} is probably caused by singularities of the eigenfunction u_1 at the obtuse angles. The strength of these singularities increases with the size of these angles, but the resolution of the used meshes stays the same.

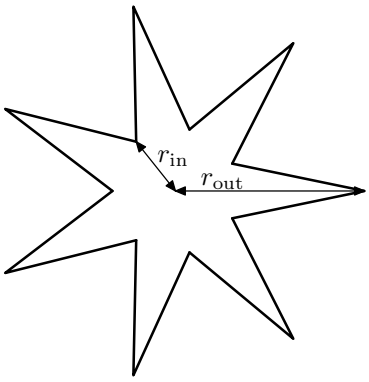

Figure 3: Illustration of 7-fold regular star with inner and outer radii.

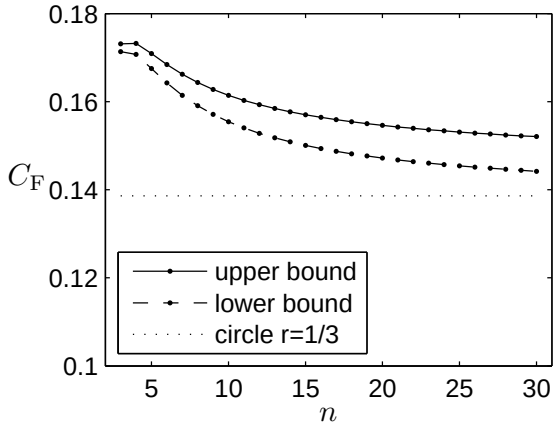

Figure 4: Values of Friedrichs' constant for n-fold regular stars.

7. Example C: Dependence on the Dirichlet part

In the final example we investigate the dependence of C_{F} on the Dirichlet part Γ_{D} of the boundary $\partial\Omega$. We consider a fixed L-shaped domain Ω and vary the position

and size of Γ_D. The boundary $\partial\Omega$ is split into 16 segments of unit length. The part Γ_D is chosen as a connected curve of length $|\Gamma_D| = \ell$, i.e. it consists of ℓ segments. In this experiment we consider $|\Gamma_D| = 1, 5, 11$, and 15. For each length we compute the lower and upper bound of C_F for all 16 positions of Γ_D on $\partial\Omega$. The positions are indexed by the number of the first segment of Γ_D in the counterclockwise sense, see Figure 5.

Figure 6 presents the dependence of C_F on the position of Γ_D for the four considered sizes $|\Gamma_D|$. We observe strong dependence both on the position and size. We also see similar values of C_F for almost symmetric positions, for instance for $|\Gamma_D| = 1$ and positions 4 and 11 or for $|\Gamma_D| = 11$ and positions 7 and 14 (these positions correspond to peeks in the graphs in both cases).

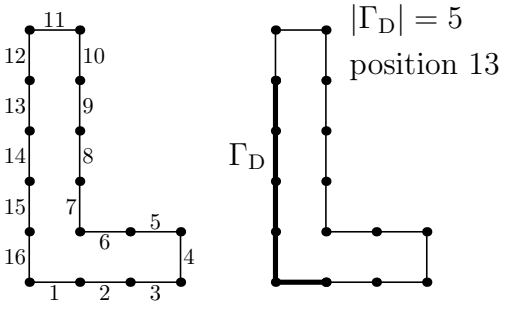

Figure 5: The L-shaped domain and enumeration of boundary segments (left). An example of a position and size of Γ_D (right).

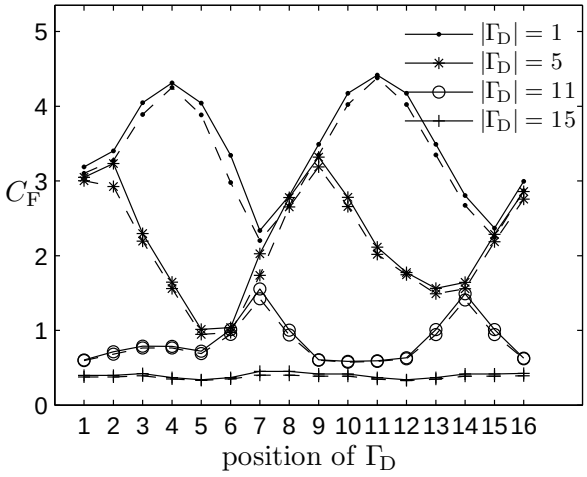

Figure 6: Dependence of Friedrichs' constant on the position and size of Γ_D. Upper bounds are indicated by solid lines and lower bounds by dashed lines.

8. Conclusions

In this contribution we present a method for computing guaranteed lower and upper bounds of Friedrichs' constant or equivalently for lower and upper bounds of eigenvalues of the corresponding differential operator. The main output is a numerical study of the value of Friedrichs' constant C_F in various cases including convex and nonconvex domains. We observe the dependence of C_F on the shape of the domain Ω and on the size and position of the Dirichlet part Γ_D of the boundary $\partial\Omega$. While we observed negligible dependence of C_F on the shape of nonobtuse triangles, the dependence on the size and position of the Dirichlet part Γ_D is significant in majority of tested cases.

Let us conclude this contribution by a note that the presented method can be easily generalized to compute two-sided bounds of the optimal constants in similar

inequalities, like the trace inequalities, Poincaré inequality, and Korn's inequality. For all these inequalities the computation of the optimal constant reduces to the computation of the smallest eigenvalue of a differential operator.

Acknowledgements

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