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HIGH RESOLUTION SCHEMES FOR OPEN CHANNEL FLOW*

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Abstract

One of the commonly used models for river flow modelling is based on the Saint-Venant equations – the system of hyperbolic equations with spatially varying flux function and a source term. We introduce finite volume methods that solve this type of balance laws efficiently and satisfy some important properties at the same time. The properties like consistency, stability and convergence are necessary for the mathematically correct solution. However, the schemes should be also positive semidefinite and preserve steady states to obtain physically relevant solution of the flow problems. These schemes can also be modified to a high order version or for solving flow problems with a friction source term.

1 Introduction

One of the most general models for simulating fluid flow is based on the Navier-Stokes equations. This model is suitable for viscous incompressible flow, but it is not directly applicable to open channel flow problems. In this case it is necessary to define some conditions on moving boundary or to use the model with the interaction between water and air layer. In our case it is convenient to use the simpler model based on the Saint-Venant equations. They are the most common choice which describes incompressible open channel flow, where vertical component of the acceleration is neglected. This model can be used for river flow or for problems of coastal areas flow.

2 Mathematical model

The one-dimensional Saint-Venant equations have the following form:

$$\begin{aligned} h_t + (hv)_x &= 0, \\ (hv)_t + \left(hv^2 + \frac{1}{2}gh^2\right)_x &= -ghB_x, \end{aligned} \tag{1}$$

where $h = h(x,t)$ is the unknown fluid depth, $v = v(x,t)$ is the unknown horizontal velocity, $B = B(x)$ is the elevation of the bottom surface and g is the gravitational constant. The other source terms (e.g. friction term important for flood modelling) can be added into the system. In the following parts of this paper we use, for

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simplicity, the system in the form (1). This system can be simply written in the matrix form

$$\mathbf{u}_t + [\mathbf{f}(\mathbf{u})]_x = \boldsymbol{\psi}(\mathbf{u}, x). \tag{2}$$

The following schemes use the finite volume discretization with the space step Δx such that $x_j = j\Delta x, j \in \mathbb{Z}$, and adaptive time step Δt_n based on the CFL stability condition.

3 Properties of the methods

In addition to important properties like conservation, consistency and stability the numerical schemes should satisfy some other ones.

- Positive semidefiniteness – some of the unknown functions have to be nonnegative from their physical fundament. Therefore it is necessary to use such a scheme that satisfies the nonnegativity of these functions. We suppose $h \geq 0$ in our problem.

- Preserving steady states – the numerical scheme should preserve such steady states, which occur in the exact solution. The steady state means $\mathbf{u}_t = \mathbf{0}$ and therefore $[\mathbf{f}(\mathbf{u}, x)]_x = \boldsymbol{\psi}(\mathbf{u}, x)$. Then the numerical scheme should balance the flux difference and the approximation of the source terms. The presented schemes do not preserve general steady states but only the special one called "rest at lake" ($h + B =$ const., $v = 0$).

- High resolution – we can construct the scheme of the high order of accuracy. However, the high order schemes produce spurious oscillations in the regions with discontinuities in the solution. Therefore our goal is to construct such a scheme which is of high order of accuracy in the area with the smooth solution and first order accurate if there exist jumps in the solution. Moreover, this scheme should contain a small amount of artificial diffusion.

Furthermore, the big advantage of the method is its possibility to use long time steps, especially if we solve large scale problems. From this point of view we can choose between explicit and implicit methods.

Explicit methods are easier to implement and they have low cost per time step, because they need not solve any system of algebraic equations. However, the time step is bounded by the CFL stability condition. Furthermore, they are often inefficient for the solution of the stationary problems.

On the other hand, the implicit methods are unconditionally stable or stable over a wide range of the time steps. But they have high cost per time step which is caused by solving the system of algebraic equations. The linear solvers have also problems with convergence as time step increases. Implicit schemes are often insufficiently accurate for transient problems at large time step.

The main idea is to construct an adaptive semi-implicit scheme with advantages of the implicit and explicit methods.

4 Semi-implicit upwind method

The following scheme is based on the Roe type scheme described in [1]. The general semi-implicit finite volume scheme for balance laws in the conservative form can be written as

$$\frac{\mathbf{U}_j^{n+1}-\mathbf{U}_j^n}{\Delta t} = -\frac{1}{\Delta x}[(1-\theta)(\mathbf{F}^n_{j+1/2}-\mathbf{F}^n_{j-1/2})+\theta(\mathbf{F}^{n+1}_{j+1/2}-\mathbf{F}^{n+1}_{j-1/2})]+(1-\theta)\mathbf{\Psi}^n_j+\theta\mathbf{\Psi}^{n+1}_j, \tag{3}$$

where $\mathbf{U}_j^n$ is the approximation of integral average of unknown function $\mathbf{u}(x,t)$ in the cell $\langle x_{j-1/2,j+1/2}\rangle$ at the time t_n

$$\mathbf{U}_j^n \approx \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} \mathbf{u}(x,t_n)dx.$$

The numerical fluxes $\mathbf{F}^n_{j+1/2}$ approximate the flux function at the the boundary of neighbouring cells j and $j+1$ and $\mathbf{\Psi}^n_j$ is a suitable approximation of the source term in the cell $\langle x_{j-1/2,j+1/2}\rangle$. The finite volume methods are in detail described in [3].

The parameter θ takes values from the interval $\langle 0,1\rangle$. For $\theta=0$ the scheme is explicit, for $\theta=1$ it is implicit and for $0<\theta<1$ it is the semi-implicit scheme. The time step of the explicit scheme for hyperbolic problems is bounded by the stability CFL condition. The CFL number can be defined as

$$\text{CFL } = \frac{\Delta t}{\Delta x}\max_{p=1,2}|\lambda^p|,$$

where λ^p are approximations of the eigenvalues of the Jacobian matrix $\partial\mathbf{f}/\partial\mathbf{u}$. It has been shown [1] that the CFL number for the Roe type semi-implicit scheme satisfies

$$\text{CFL } \le \frac{1}{1-\theta}$$

in the scalar case. The construction of the numerical fluxes at the time level t_n is based on the approximate Jacobian matrix $\mathbf{A}^n_{j+1/2}\approx \partial\mathbf{f}/\partial\mathbf{u}(x_{j+1/2},t_n)$. The numerical flux has the form:

$$\mathbf{F}^n_{j+1/2} = \frac{1}{2}[\mathbf{f}(\mathbf{U}^n_j)+\mathbf{f}(\mathbf{U}^n_{j+1})] - \frac{1}{2}|\mathbf{A}^n_{j+1/2}|(\mathbf{U}^n_{j+1}-\mathbf{U}^n_j),$$

where

$$|\mathbf{A}^n_{j+1/2}| = \mathbf{R}^n_{j+1/2}|\mathbf{\Lambda}^n_{j+1/2}|\mathbf{L}^n_{j+1/2}\mathbf{R}^n_{j+1/2}.$$

Here, $|\mathbf{\Lambda}^n_{j+1/2}| = \text{diag}(|\lambda^{p,n}_{j+1/2}|)$, where $\lambda^{p,n}_{j+1/2}$ are eigenvalues of $\mathbf{A}^n_{j+1/2}$, and $\mathbf{R}^n_{j+1/2}$ is the matrix of the right eigenvectors of $\mathbf{A}^n_{j+1/2}$. In the case of the first order scheme the matrix $\mathbf{L}^n_{j+1/2}$ is the identity matrix $\mathbf{I}$, in the case of the flux limited scheme it has the form

$$\mathbf{L}^n_{j+1/2} = \mathbf{I} + \text{ diag}\left(\varphi(\mathbf{u})\left(1-\min\left\{1, |\lambda^{p,n}_{j+1/2}|\frac{\Delta t}{\Delta x}\right\}\right)\right),$$

where $\varphi(\mathbf{u})$ is some limiter function based on the jumps of the unknown function $\mathbf{u}(x,t)$. It is clear that for CFL >1 we also obtain the first order upwind scheme.

The construction of the numerical fluxes at the time level t_{n+1} is very similar. We use new values of the unknown function $\mathbf{U}^{n+1}$, but if we do not want to solve a nonlinear system of algebraic equations it is necessary to use a linearization for evaluating the flux function, i.e.

$$\mathbf{f}(\mathbf{U}_j^{n+1}) \approx \mathbf{f}(\mathbf{U}_j^n) + \mathbf{A}_{j+1/2}^n(\mathbf{U}_j^{n+1} - \mathbf{U}_j^n). \tag{4}$$

It remains to define the approximation of the source terms. To preserve the balancing property it is useful to decompose the source term integral in a similar way as the numerical fluxes:

$$\mathbf{\Psi}_j^n = \mathbf{\Psi}_{j+1/2}^{n,-} + \mathbf{\Psi}_{j-1/2}^{n,+},$$

where

$$\mathbf{\Psi}_{j+1/2}^{n,\pm} = \frac{1}{2}(\mathbf{I} \pm \mathbf{A}_{j+1/2}^{-1}|\mathbf{A}_{j+1/2}|)\mathbf{\Psi}_{j+1/2}^n.$$

Then we can construct a block tridiagonal system of the linear equations.

5 Semi-implicit central-upwind method

Central-upwind schemes, based on the scheme described in [2], preserve only special steady states, where the spatial derivatives of unknown functions (or their reconstructions) are equal to zero. So we define new unknown function for water level $c = h + B$ (to preserve special steady state "rest at lake", where $hv = 0$ and $c = h + B =$ const.). Then the system of the Saint-Venant equations can be rewritten in terms of c and momentum hv as

$$\begin{pmatrix} c \\ hv \end{pmatrix}_t + \begin{pmatrix} hv \\ (hv)^2/(c-B) + g(c-B)^2/2 \end{pmatrix}_x = \begin{pmatrix} 0 \\ -g(c-B)B_x \end{pmatrix}.$$

The semidiscrete conservative scheme has the following form:

$$\frac{d}{dt}\mathbf{U}_j(t) = -\frac{\mathbf{F}_{j+1/2}(t) - \mathbf{F}_{j-1/2}(t)}{\Delta x} + \mathbf{\Psi}_j(t).$$

Numerical fluxes at the time t_n are defined as (see [2])

$$\mathbf{F}_{j+1/2}^n = \frac{a_{j+1/2}^{n,+}\mathbf{f}(\mathbf{U}_{j+1/2}^{n,-}) - a_{j+1/2}^{n,-}\mathbf{f}(\mathbf{U}_{j+1/2}^{n,+})}{a_{j+1/2}^{n,+} - a_{j+1/2}^{n,-}} + \frac{a_{j+1/2}^{n,+}a_{j+1/2}^{n,-}}{a_{j+1/2}^{n,+} - a_{j+1/2}^{n,-}}\left[\mathbf{U}_{j+1/2}^{n,+} - \mathbf{U}_{j+1/2}^{n,-}\right], \tag{5}$$

where the approximations of the speeds of the local waves are defined as

$$\begin{aligned} a_{j+1/2}^{n,+} &= \max\left\{\lambda^2\left(\mathbf{f}'(\mathbf{U}_{j+1/2}^{n,-})\right), \lambda^2\left(\mathbf{f}'(\mathbf{U}_{j+1/2}^{n,+})\right), 0\right\}, \\ a_{j+1/2}^{n,-} &= \min\left\{\lambda^1\left(\mathbf{f}'(\mathbf{U}_{j+1/2}^{n,-})\right), \lambda^1\left(\mathbf{f}'(\mathbf{U}_{j+1/2}^{n,+})\right), 0\right\}, \end{aligned} \tag{6}$$

and $\mathbf{U}^{n,\pm}_{j+1/2}$ are the left and the right values of some polynomial reconstruction of the unknown function at $x_{j+1/2}$ (in this case $\mathbf{U}^{n,\pm}_{j+1/2} = [C^{n,\pm}_{j+1/2}, (HV)^{n,\pm}_{j+1/2}]^T$). There exist many available reconstructions. We use the following polynomial TVD reconstruction (the symbol U represents the components of the vector $\mathbf{U}$):

$$\begin{aligned} U^{n,+}_{j+1/2} &= U^n_{j+1} - \left(1 - \min\left\{1, \lambda^{\max}_{j+1/2}\tfrac{\Delta t}{\Delta x}\right\}\right)\tfrac{\Delta x}{2}(U_x)^n_{j+1}, \\ U^{n,-}_{j+1/2} &= U^n_j + \left(1 - \min\left\{1, \lambda^{\max}_{j+1/2}\tfrac{\Delta t}{\Delta x}\right\}\right)\tfrac{\Delta x}{2}(U_x)^n_j, \end{aligned} \tag{7}$$

where $\lambda^{\max}_{j+1/2} = \max\limits_{p=1,2}|\lambda^{p,n}_{j+1/2}|$ and the symbol $(U_x)^n_j$ stands for

$$(U_x)^n_j = \begin{cases} (U_x)^n_{j,L} & \text{if } |(U_x)^n_{j,L}| \le |(U_x)^n_{j,R}| \text{ and } (U_x)^n_{j,L}\cdot(U_x)^n_{j,R} > 0, \\ (U_x)^n_{j,R} & \text{if } |(U_x)^n_{j,L}| > |(U_x)^n_{j,R}| \text{ and } (U_x)^n_{j,L}\cdot(U_x)^n_{j,R} > 0, \\ 0 & \text{if } (U_x)^n_{j,L}\cdot(U_x)^n_{j,R} \le 0, \end{cases} \tag{8}$$

where

$$(U_x)^n_{j,L} = \frac{U^n_j - U^n_{j-1}}{\Delta x}, \qquad (U_x)^n_{j,R} = \frac{U^n_{j+1} - U^n_j}{\Delta x}.$$

To preserve the special steady state "rest at lake" it is also necessary to choose approximation of the source term which is equal to the numerical flux difference. This difference can be expressed as

$$\begin{aligned} -\frac{F^{n,(2)}_{j+1/2} - F^{n,(2)}_{j-1/2}}{\Delta x} &= -\frac{1}{2\Delta x}g\left(\left(C^n_{j+1/2} - B(x_{j+1/2})\right)^2 - \left(C^n_{j-1/2} - B(x_{j-1/2})\right)^2\right) \\ &= g\frac{B(x_{j+1/2}) - B(x_{j-1/2})}{\Delta x}\cdot\frac{C^n_{j+1/2} - B(x_{j+1/2}) + C^n_{j-1/2} - B(x_{j-1/2})}{2}. \end{aligned}$$

Therefore the consistent discretization of the source terms has the form

$$\Psi^{n,(2)}_j = -g\frac{B(x_{j+1/2}) - B(x_{j-1/2})}{\Delta x}\cdot\frac{\left(C^{n,-}_{j+1/2} - B(x_{j+1/2})\right) + \left(C^{n,+}_{j-1/2} - B(x_{j-1/2})\right)}{2}. \tag{9}$$

Now we are ready to construct the semi-implicit central-upwind scheme based on the same ideas as the semi-implicit upwind scheme described before. This scheme has the form (3) and the numerical fluxes at the time level t_{n+1} are defined as follows:

$$\mathbf{F}^{n+1}_{j+1/2} = \frac{a^{n,+}_{j+1/2}\mathbf{f}(\mathbf{U}^{n+1,-}_{j+1/2}) - a^{n,-}_{j+1/2}\mathbf{f}(\mathbf{U}^{n+1,+}_{j+1/2})}{a^{n,+}_{j+1/2} - a^{n,-}_{j+1/2}} + \frac{a^{n,+}_{j+1/2}a^{n,-}_{j+1/2}}{a^{n,+}_{j+1/2} - a^{n,-}_{j+1/2}}\left[\mathbf{U}^{n+1,+}_{j+1/2} - \mathbf{U}^{n+1,-}_{j+1/2}\right]. \tag{10}$$

We can see that the approximations of the maximum speeds of the local wave are the same as the approximations at the time level t_n. The reconstruction of the unknown functions is based on the (7) again. However, if we use (7) (especially choice of the differences (8)) for the values at the time level t_{n+1} by the same way as for the values at the time level t_n then (3) is the nonlinear system of algebraic equations. Therefore we define the components of $(\mathbf{U}_x)_j^{n+1}$ as

$$(U_x)_j^{n+1} = \begin{cases} (U_x)_{j,L}^{n+1} & \text{if } |(U_x)_{j,L}^n| \le |(U_x)_{j,R}^n| \text{ and } (U_x)_{j,L}^n \cdot (U_x)_{j,R}^n > 0, \\ (U_x)_{j,R}^{n+1} & \text{if } |(U_x)_{j,L}^n| > |(U_x)_{j,R}^n| \text{ and } (U_x)_{j,L}^n \cdot (U_x)_{j,R}^n > 0, \\ 0 & \text{if } (U_x)_{j,L}^n \cdot (U_x)_{j,R}^n \le 0, \end{cases}$$

where

$$(U_x)_{j,L}^{n+1} = \frac{U_j^{n+1} - U_{j-1}^{n+1}}{\Delta x}, \qquad (U_x)_{j,R}^{n+1} = \frac{U_{j+1}^{n+1} - U_j^{n+1}}{\Delta x}.$$

Then (3) is the linear system of algebraic equations. If we use CFL > 1, the reconstructed function is piecewise constant and the scheme is of the first order of accuracy.

The linearization of the flux function is provided in the same manner as in (4). The approximation of the source terms is simply defined by (9) with the reconstruction values $C^{n+1,\pm}$, i.e.

$$\Psi_j^{n+1,(2)} = -g\frac{B(x_{j+1/2}) - B(x_{j-1/2})}{\Delta x} \cdot \frac{\left(C_{j+1/2}^{n+1,-} - B(x_{j+1/2})\right) + \left(C_{j-1/2}^{n+1,+} - B(x_{j-1/2})\right)}{2}.$$

and the scheme still preserves the steady state "rest at lake".

6 Numerical experiment

This experiment simulates the steady state "rest at lake". The described variants of the central-upwind method are used. The initial conditions (Figure 1, top left) are defined by

$$h(x, 0) + B(x) = 12, \qquad v(x, 0) = 0.$$

Boundary conditions are defined by zero discharge $q(0, t) = \text{const.} = 0$ and extrapolation of water level at the left boundary. The extrapolation of the discharge and water level is used on the right end of the interval. In Figure 1 we can see the comparison between the solutions computed by the balanced (bottom left) and unbalanced (top right) explicit method. In the case of balanced implicit method (bottom right) CFL $= 1000$ is used and the solution is depicted at the time $t = 10000s$.

7 Conclusions

We presented the high-resolution semi-implicit central upwind scheme for solving the Saint-Venant equations, which combines some of the advantages of implicit and explicit methods. As the basis for the implicit method we used the explicit method, which is positive, computationally efficient and preserves the special steady states. Since the method is nonlinear due to nonlinearity of the problem and the use of the limiter, we proposed the special linearized reconstruction of unknown functions at the time level t_{n+1}. The resulting semi-implicit method preserves the special steady states and it is also positive.

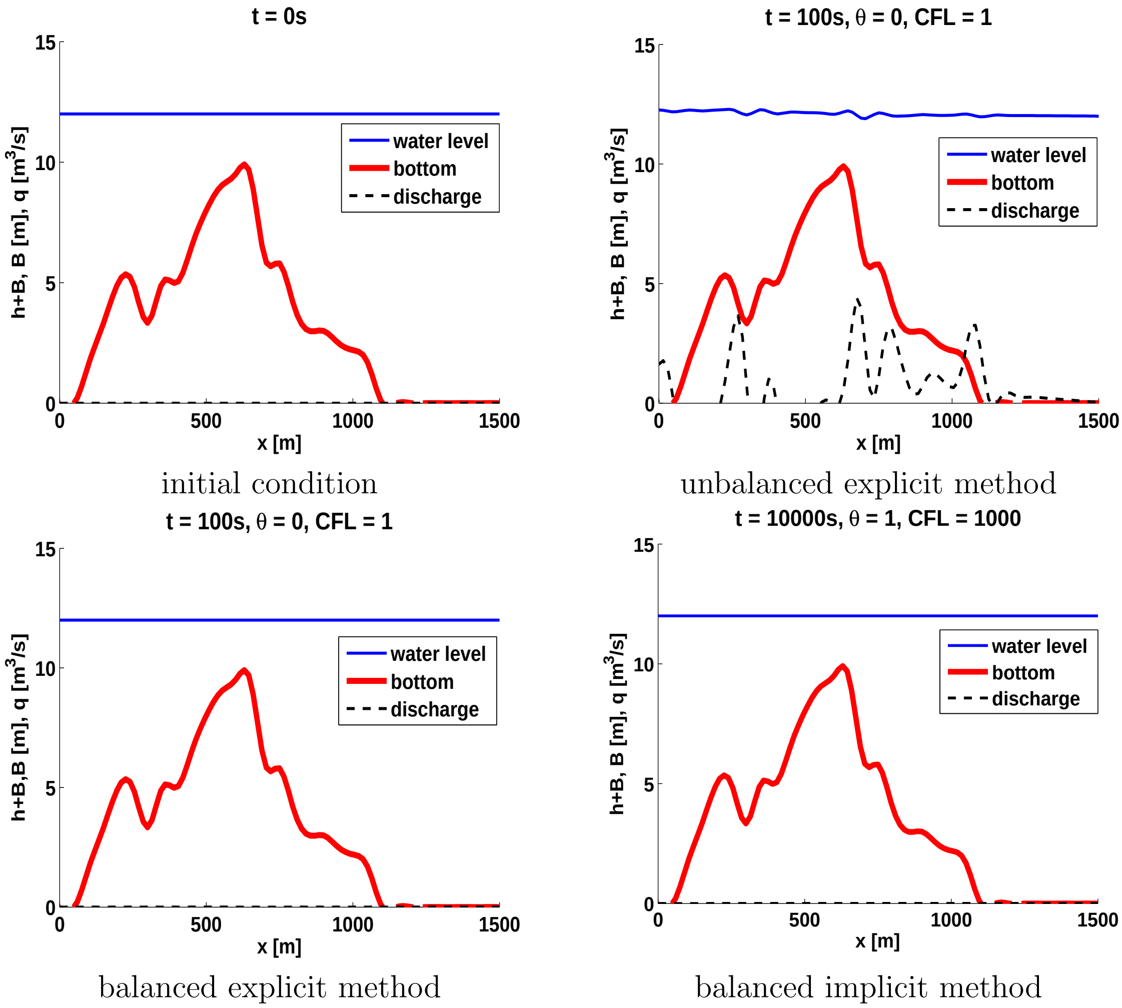

Fig. 1: *Comparison of the approximate solutions for the steady state problem.*

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