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NUMERICAL INTEGRATION IN THE TREFFTZ FINITE ELEMENT METHOD

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Abstract: Using the high order Trefftz finite element method for solving partial differential equation requires numerical integration of oscillating functions. This integration could be performed, instead of classic techniques, also by the Levin method with some modifications. This paper shortly describes both the Trefftz method and the Levin method with its modification.

Keywords: Trefftz finite element method, Levin method

MSC: 65N30, 65D30

1. Introduction

The Trefftz finite element method is a method for solving boundary value problems applied already in the 1970. The integrals of oscillating functions often appear during the solving procedure. These integrals can be calculated using the modified Levin method. In this paper we briefly introduce the Trefftz and Levin methods with its necessary modification to be applicable to the Trefftz finite element.

2. Trefftz finite element method

Trefftz finite elements are finite elements based on the usage of the auxiliary unknown defined on the edges (faces) of elements that links together the primary unknown defined on each element. This method is described in [1] or [5]. Description of implementation aspect is in [6].

Here, let us briefly present the method on the following model problem. Consider that we are seeking to find the solution of the Laplace equation in a domain $\Omega \subset \mathbb{R}^2$ endowed with the boundary conditions

$$\begin{aligned} \Delta u &= 0 \quad \text{in} \quad \Omega, \\ u &= \bar{u} \quad \text{on} \quad \Gamma_u, \\ \frac{\partial u}{\partial n} &= \bar{q} \quad \text{on} \quad \Gamma_q, \end{aligned} \tag{1}$$

where $\bar{u}, \bar{q}$ are known functions, and n is the normal to the boundary $\Gamma = \Gamma_u \cap \Gamma_q$.

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Let the domain Ω be divided into elements; and over each element Ω_e, we assume the function u in the form

$$u = \sum_{j=1}^{M} c_j \, N_j \quad \text{on element } \Omega_e,$$

where c_j are unknown constants and N_j are known functions to be chosen such that

$$\Delta N_j = 0 \quad \text{in element } \Omega_e, \text{ for } j = 1, 2, \ldots, M.$$

It can be shown that this equation is satisfied by any of the following functions:

$$1,\, r\cos\theta,\, r\sin\theta, \ldots,\, r^m \cos m\theta,\, r^m \sin m\theta, \ldots, \tag{2}$$

where r and θ are a pair of polar coordinates.

Let us introduce an auxiliary function $\tilde{u}$ defined on element boundary only:

$$\tilde{u} = \sum_{i=1}^{N} d_i \, \tilde{N}_i,$$

where d_i stands for nodal displacement and $\tilde{N}_i$ are standard shape functions.

Let us denote $q = \frac{\partial u}{\partial n}$ and $q_1 = \frac{\partial u}{\partial x_1}, q_2 = \frac{\partial u}{\partial x_2}$. Following the approach in [5], let us introduce the functional Ψ_e,

$$\Psi_e = \frac{1}{2} \int_{\Omega_e} q_1^2 + q_2^2 \,\mathrm{d}\Omega - \int_{\Gamma_e} q \, \tilde{u} \,\mathrm{d}\Gamma + \int_{\Gamma_{eq}} \bar{q} \, \tilde{u} \,\mathrm{d}\Gamma = \frac{1}{2} \int_{\Gamma_e} q \, u \,\mathrm{d}\Gamma - \int_{\Gamma_e} q \, \tilde{u} \,\mathrm{d}\Gamma + \int_{\Gamma_{eq}} \bar{q} \, \tilde{u} \,\mathrm{d}\Gamma, \tag{3}$$

where Γ_e is the boundary of the element Ω_e and $\Gamma_{eq} = \Gamma_e \cap \Gamma_q$. The minimalization of the variational functional Ψ_e for the all elements provides the solution of (1).

3. Levin method

The Levin method is an effective way for the numerical integration of rapidly oscillating functions. It is described in [2] for one- and two-dimensional integrals; more details, numerical examples, and error analysis are provided in [3], [4].

Let us briefly introduce this method. The integration problem is transformed into an ordinary differential equation problem to be numerically solved by for example the collocation method.

We consider integrals of the form

$$I = \int_a^b f^t \cdot w \,\mathrm{d}x, \tag{4}$$

where f is a vector of smooth and non-oscillating functions, $f(x) = (f_1(x), \ldots, f_m(x))^t$, w is a vector of oscillating functions, $w(x) = (w_1(x), w_2(x), \ldots w_m(x))^t$, and a, b are real and finite. We also assume that w satisfies

$$w'(x) = A(x)\, w(x),$$

where A is an $m \times m$ matrix of non-oscillating functions.

We would like to find the vector $p(x) = (p_1(x), p_2(x), \ldots p_m(x))^t$ such that

$$(p^t \cdot w)' = f^t \cdot w.$$

Subsequently,

$$I = \int_a^b \left(p^t \cdot w\right)' \mathrm{d}x = \int_a^b (p')^t \cdot w + p^t \cdot w' \,\mathrm{d}x = \int_a^b (p')^t \cdot w + p^t \cdot A\, w \,\mathrm{d}x = \int_a^b \left(p' + A^t\, p\right)^t \cdot w \,\mathrm{d}x,$$

Hence, the vector p should satisfy

$$p' + A^t\, p = f.$$

Then, the integral is computed as

$$\int_a^b f^t \cdot w \,\mathrm{d}x = p^t(b) \cdot w(b) - p^t(a) \cdot w(a). \tag{5}$$

Example As an example, let us compute the integral

$$\int_0^{2\pi} x^2 \cos(rx) \,\mathrm{d}x = \int_0^{2\pi} \left(x^2, 0\right) \cdot (\cos(rx), \sin(rx)) \,\mathrm{d}x,$$

where $r \in \odot \mathbb{N}$. In the notation used above, $f(x) = \left(x^2, 0\right)^t$ and $w(x) = (\cos(rx), \sin(rx))^t$.

Then,

$$w'(x) = \begin{bmatrix} \cos(rx) \\ \sin(rx) \end{bmatrix}' = \begin{bmatrix} 0 & -r \\ r & 0 \end{bmatrix} \begin{bmatrix} \cos(rx) \\ \sin(rx) \end{bmatrix} = A(x)\, w(x).$$

We are looking for the vector $p = (p_1, p_2)$ which satisfies

$$\begin{aligned} p_1' + r p_2 &= x^2, \\ p_2' - r p_1 &= 0. \end{aligned}$$

The general solutions are

$$p_1(x) = C_1 \cos(rx) + C_2 \sin(rx) + \frac{2x}{r^2}, \tag{6}$$

$$p_2(x) = C_1 \sin(rx) - C_2 \cos(rx) + \frac{x^2}{r} - \frac{2}{r^3}. \tag{7}$$

The formula (5) is valid for any solution of (6). We choose the solution for which $C_1 = C_2 = 0$. Hence,

$$\int_0^{2\pi} x^2 \cos(rx)\, \mathrm{d}x = \left[\left(\frac{2x}{r^2}, \frac{x^2}{r} - \frac{2}{r^3} \right) \cdot (\cos(rx), \sin(rx))^t \right]_0^{2\pi} = \frac{4\pi}{r^2}.$$

4. Integration in the Trefftz method

Using functions (2) in functional (3) leads to line integrals of oscillating functions.

It depends on the implementation, but it is usual that only values of the integrated function are accessible. In this content, it is obvious that success of Levin method relies on rewriting w' as $A\,w$. In this situation, namely, when only values of w are known, finding matrix A could be a problem.

The oscillating function w can be approximated by the trigonometric interpolation polynomial

$$w(x) \doteq a_0 + \sum_{i=1}^{n} \left(a_i \cos(\alpha_i x) + b_i \sin(\alpha_i x) \right),$$

where the coefficients a_0, a_i, b_i, $i = 1 \dots N$, can be effectively computed by the discrete fast Fourier transform.

Then, the integral of the form (4) can be approximated by

$$I \doteq \int_a^b f\, a_0 + \sum_{i=1}^{n} f\, \left(a_i \cos(\alpha_i x) + b_i \sin(\alpha_i x) \right) \mathrm{d}x =$$

$$a_0 \int_a^b f\, \mathrm{d}x + \int_a^b \sum_{i=1}^{n} f\, \left(a_i \cos(\alpha_i x) + b_i \sin(\alpha_i x) \right) \mathrm{d}x.$$

In this form, the integral is suitable for the Levin method.

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