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STRATEGIES FOR COMPUTATION OF LYAPUNOV EXPONENTS ESTIMATES FROM DISCRETE DATA

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Abstract: The Lyapunov exponents (LE) provide a simple numerical measure of the sensitive dependence of the dynamical system on initial conditions. The positive LE in dissipative systems is often regarded as an indicator of the occurrence of deterministic chaos. However, the values of LE can also help to assess stability of particular solution branches of dynamical systems. The contribution brings a short review of two methods for estimation of the largest LE from discrete data series. Two methods are analysed and their freely available Matlab implementations are tested using two sets of discrete data: the sampled series of the Lorenz system and the experimental record of the movement of a heavy ball in a spherical cavity. It appears that the most important factor in LE estimation from discrete data series is quality of the available record.

Keywords: dynamical system, Lyapunov exponents, stability, data analysis

MSC: 37M10, 37M25, 34C25, 34D30

1. Introduction

Vibration or general movement analysis of non-linear structures, signal analysis of electrical circuits, etc., represent a challenging task in various branches of engineering. This regards the both cases of mathematical and experimental models or an analysis of results from measurements in situ. In case of a complex behaviour of a structure or a non-linear mathematical model, the measured response or computed data series can exhibit a wide range of response types, from stationary and periodic to diverging or chaotic behaviour. The stability in the sense of sensitivity to small perturbations, however, is the key property of each type of the system response.

Even if the topic was addressed by numerous papers in the past it seems that the practical usage of recommended methods usually raise additional questions. Indeed, the theoretical results are mostly substantiated by a limiting relation whose assumptions are hardly fulfilled in the practice.

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It also seems that the concept of Lyapunov exponents (LE) is still the most usable and most robust stability measure, despite of numerous new methods and modifications. It often becomes apparent that the new methods designed for stability assessment are closely related to LE, either representing a special case or generalization of the usual approach, see, e.g., [4].

Implementation and performance of several algorithms regarding continuous systems is reviewed in [7]. A very promising approach for continuous systems is presented by Dieci et al. in [2], where a possible extension to certain discrete cases is briefly mentioned. This contribution, however, extends the previous work [3] and reviews two classical approaches convenient for cases with discrete data series.

2. Lyapunov exponents

Let us consider the n dimensional non-linear differential initial value problem

$$x'(x_0, t) = f(x, t)\,, \qquad x(x_0, 0) = x_0, \tag{1}$$

where $'$ means derivation with respect to t. The right-hand side f is supposed to be a smooth function and the bounded solution $x(x_0, t)$ continuously dependent on the initial value x_0 and $t > 0$. Let $\delta_0 = \delta(0)$ be a small perturbation to the initial condition x_0. Behaviour of the perturbed trajectory follows from the values of the right-hand side f for the perturbed argument

$$f(x + \delta) \approx f(x) + \mathbf{J}_f(x)\delta, \tag{2}$$

where $\mathbf{J}_f$ denotes the Jacobian of the right-hand side f. Then evolution of the distance between original and perturbed trajectories,

$$\delta(t) = x(x_0, t) - x(x_0 + \delta(0), t) \tag{3}$$

can be described by the linearized equation

$$\delta'(t) = \mathbf{J}_f(x(x_0, t), t)\,\delta(t). \tag{4}$$

Evolution of size of the perturbation δ is governed by the relation

$$||\delta(t)|| = \mathrm{e}^{\lambda_1 t}||\delta(0)|| \tag{5}$$

and λ_1 is denoted to be the largest LE. The relation (5) is usually supposed to serve as a formula for calculation of λ_1:

$$\lambda_1 = \lim_{t\to\infty} \frac{1}{t} \ln ||\delta(t)||. \tag{6}$$

Certain similarity of the LE estimation and computation of eigenvalues of the Jacobian is apparent from (4), (5). This similarity is preserved if several or all LE are taken into account. The general setting works with so-called Lyapunov spectral intervals, for details see [2] and literature cited there. In so called *regular* systems the spectral intervals degenerate to simple values of LE. This property is usually assumed for simplification and also this work will follow this practice.

Definition 1. *Let* $A(t) = \mathbf{J}_f(x(x_0,t),t)$ *be the Jacobian of the system (1) along the trajectory* $x(x_0,t)$*,* $\mathbf{Y}(t)$ *is the* $n \times n$ *matrix and* λ_i*,* $i = 1,\dots,n$ *are numbers such that*

$$\mathbf{Y}'(t) = A(t)\mathbf{Y}(t)\,, \quad \mathbf{Y}(0) = \mathbf{Y}_0\,, \quad \mathbf{Y}_0 \in \mathbb{R}^{n\times n} \text{ regular,} \tag{7}$$

$$\lambda_i = \limsup_{t\to\infty} \frac{1}{t} \ln ||\mathbf{Y}(t)e_i|| \tag{8}$$

where e_i *is the* i*-th vector of the standard basis. Such* λ_i *which minimize* $\sum_{i=1}^{n} \lambda_i$ *for all possible initial conditions* $\mathbf{Y}_0$ *are called Lyapunov exponents.*

Regularity of the underlying system is an important property, which enables validity of the standard computational rule (6).

3. Estimates of Lyapunov exponents

The commonly used estimates of LE fall into two main categories. The first uses a heuristic approach based on equation (6) which measures the divergence of close orbits. Although use of this approach is not limited to the cases where only the discrete data are available, these methods are used mostly in such cases. The second group, on the other hand, is based on relation (1) and exploits knowledge of the Jacobian. Thus, it is naturally aimed at analysis of continuous systems. This division is natural and dates back to first papers on the topic, cf. the paper due to Wolf et al. [9].

3.1. Estimates exploiting Jacobian

Consider solution $\tilde{x}_{x_0}$ to the continuous dynamical system (1). The solution continuously depends on the initial condition. Differentiating (1) with respect to x_0 one gets the *variational equation*, see [6]:

$$\mathbf{P}'(t) = A(x,t)\mathbf{P}(t)\,, \qquad \mathbf{P}(0) = \mathbf{I}, \tag{9}$$

where $\mathbf{P}(t) = \partial_{x_0}(\tilde{x}'(t))$ and $A(\tilde{x},t) = \partial_{x_0} f(\tilde{x}(t),t)$. The system (1) can be regarded w.l.o.g. as autonomous, then $A(x,t)$ is the Jacobian of f and evolution of an initial perturbation δ_0 of (3) satisfies

$$\delta(t) = \mathbf{P}(t)\delta_0.$$

The last relation closely corresponds to equations (4) and (7) and is a basis of a large family of procedures aimed at estimation of LEs of a continuous system. Their common problem is the necessity to keep the matrix $\mathbf{P}(t)$ orthogonal. A promising implementation is presented in [2]. In the accompanying documentation Dieci et al. claim applicability to the cases with discrete data, provided that the data are continuously interpolated. Knowledge of the Jacobian along the trajectory or at least its estimate is still necessary. However, validation of this interesting option is out of scope of the present contribution.

3.2. Estimates based on trajectory

The approach based on equation (6) uses only values of close trajectories to measure evolution of a small perturbation. This approach is used mostly in the case of discrete maps or when only some measured data are available. Basic properties and weaknesses of the direct application of (6) are discussed in the short review paper by the authors [3].

The implementation of a more advanced algorithm for the discrete data which accompanies the paper due to Wolf et al. [9] gained high popularity. The code was ported to Matlab and supplemented with highly instructional notes recently [10]. However, it has some limitations, namely it requires user interaction for selected problems. The algorithm follows the nature of the problem: it is based on identification of close points on the orbit whose temporal separation in the original time series is at least one mean orbital period. Such points are considered as close (perturbed) initial conditions and separation of corresponding orbital sections is measured. The largest LE λ_1 is computed from the growth of distance of both orbits. When the separation becomes large, a new trajectory is chosen near the reference trajectory considering close distance and direction. The separation direction approximates direction of the first principal axis or first Lyapunov direction which correspond to the Jacobian matrix at each point of interest along the flow. As a next step, the algorithm allows for estimation of $\lambda_1 + \lambda_2$ from a growth of area elements, etc.

The more advanced procedure described by Rosenstein et al. [8] is similarly based on identifying different yet similar sections in the experimental data series, which are used subsequently to simulate separation of close orbits. The result of the procedure is given as the dependence of averaged distance of two trajectories on increasing time lag to initial "close" point. Theoretically, according to (5), the dependence in the logarithmic scale should increase linearly up to the size of the attractor. The slope of the linear ramp then represents an estimate of the largest LE. For any larger time lag it should attain constant values. For details, see the original reference or the short review [3].

The so called dimensional reconstruction is an important part of both algorithms in case when the experimental data consist of time series from a single observable and the underlying physical system is not known. It is usually implemented using the method of delays where the reconstructed dimension m should be $m > 2n$ and n is dimension of the (assumed) physical system. For an N-point time series x_i, $i = 1, \ldots, N$ the i-th reconstructed observation (vector) is given as

$$\mathbf{x}_i = (x_i, x_{i+\Delta}, \ldots, x_{i+(m-1)\Delta}) \tag{10}$$

where Δ is the reconstruction delay. Rosenstein et al. [8] claim that their algorithm often works well even when m is below $2n$. Regarding the value of delay Δ, Rosenstein proposes the value based on the autocorrelation function. They report the best value of Δ to equal the lag where the autocorrelation function drops to $1-1/\mathrm{e}$ of its initial value.

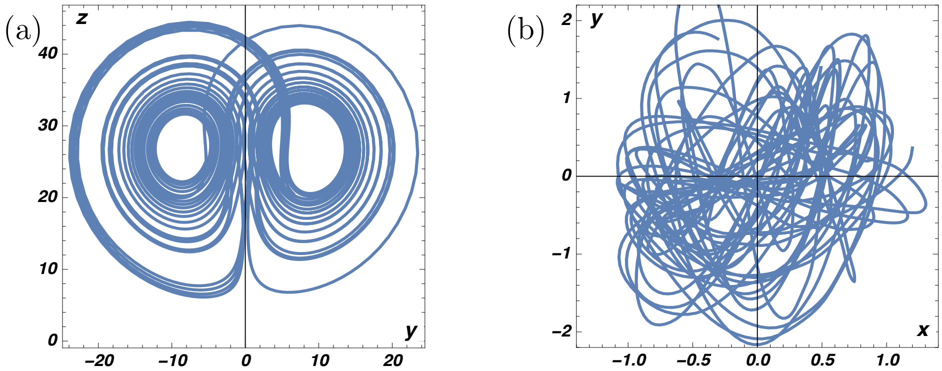

Figure 1: Sample trajectories: (a) phase plot of y, z components of the Lorenz system ($\sigma = 16, \varrho = 45.92, \beta = 4$), (b) measured displacement data of the relative motion of the ball in the cavity. In the both cases $\Delta t = 0.01$s.

4. Numerical evaluation

The two mentioned algorithms were tested using the freely available implementations in Matlab: a) the Matlab adaptation of the original FORTRAN code for the algorithm due to Wolf [10], and b) the loose implementation of the Rosenstein's algorithm by an anonymous contributor to the MathWorks site, [1].

Two sample discrete data sets were chosen, both representing continuous dynamical systems. The first one is the trajectory of the well-known Lorenz system ($\sigma = 16, \varrho = 45.92, \beta = 4$) for $t \in \langle 0, 100\rangle$, sampling period $\Delta t = 0.01$, with initial condition $\{x, y, z\} = \{0, 1, 0\}$, see Fig. 1a. First 1000 samples were discarded to assure that the trajectory belongs to the attractor, resulting to record size of (9000×3) samples. Only the x component was used to test the dimensional reconstruction.

The second example represents the spatial movement of a heavy cast iron ball freely rolling in a spherical cavity. The cavity was forced to move harmonically in one direction with a fixed frequency and amplitude. The spatial movement then arises for a critical frequency due to the non-linear character of the set-up, for details see [5] and references cited there. The recorded data consist of 3 072 samples for longitudinal (x) and transversal (y) directions, sampled at 100 Hz, see Fig. 1b. The underlying theoretical model is defined for 4 unknowns (displacements and velocities), however, the velocities can be deduced from the measured positions with reasonable accuracy.

The both data sets represent certain unfavourable properties. In the case of the Lorenz system, the attractor consists from two lobes and the trajectories digress from one to another. This behaviour prevents the Wolf's method to work autonomously and implies some interaction during evaluation, for details see [9]. Thus, Table 1 refers only value from the paper. The experimental data suffer from two different weaknesses: the record is relatively short and trajectories are scattered. On the other hand, the both underlying continuous systems are known to be chaotic; dominant LE

algorithm		Wolf	Rosenstein		
data dimension		1	1 (x)	1 (x)	3 (x, y, z)
reconstructed dimension		1	2	3	$-^{\ddagger}$
Lorenz system	λ_1 :	$2.16^{\star}$	1.82	1.71	2.135
data dimension		1	1 (x)	2 (x, y)	4 ($x, \dot{x}, y, \dot{y}$)
reconstructed dimension		1	4	4	$-^{\ddagger}$
experimental data	λ_1 :	$-^{\dagger}$	9.9	18.06	0.89

notes: *value from [9] † no convergence ‡dimensional reconstruction was not used

Table 1: Estimated values of Lyapunov exponents in bits.s^{-1}

of the selected configuration of the continuous Lorenz system computed according to [2] is 2.164 bits.s^{-1}, detailed properties of the ball moving in the spherical cavity are going to be published by the authors of this study in a near future.

The results obtained during evaluation of both algorithms are summarized in Table 1. In agreement with the paper [9], the binary logarithm was used resulting in values which can be interpreted as a loss of bits (accuracy) per second. The following cases were tested: one or two-dimensional signal with dimensional reconstruction and the fully dimensional data without reconstruction. Unfortunately, the algorithm proposed by Wolf appeared not applicable due to character of selected test data. Namely in the case of experimental data the procedure did not converged sufficiently.

The weak point of the Rosenstein's approach is the final evaluation of the computed dependence of separation of close trajectories on time, see description in [3]. The Matlab implementation [1] selects rigidly samples `15:78` for linear regression and LE estimation. These values are most probably relict of a particular study and has to be changed for other cases. The authors used a simple detection of the "corner sample" based on the horizontal direction of the upper plateau. The points on the left of the corner sample are then used to estimate the LE. A number of alternative approaches could be proposed; however, they mostly require some ad hoc intervention. The problem can be seen in Figure 2 for cases dealt in this study.

The results obtained using the Rosenstein's approach are consistent in the case of the Lorenz system, see first data row in Table 1 and left column in Figure 2, cases (a–c). The obtained values even for the one-dimensional test case and both dimensional reconstructions ($m = 2, 3$) represent reasonably accurate estimates of the dominant LE. When fully dimensional data are used, the obtained value is sufficiently close to the desired value 2.164 bits.s^{-1}. Plots (a–c) in Figure 2 show the exemplary behaviour of the trajectory separation characterized by the linear ramp and flat plateau.

Low quality of the experimental data reflects in the results presented in the second data row of Table 1. High variance of the obtained values is probably inevitable. Also plots (d–f) in Figure 2 illustrate unclear position of the "corner sample". Namely in the case (f), when full 4-dimensional data are used the two linear sections can be

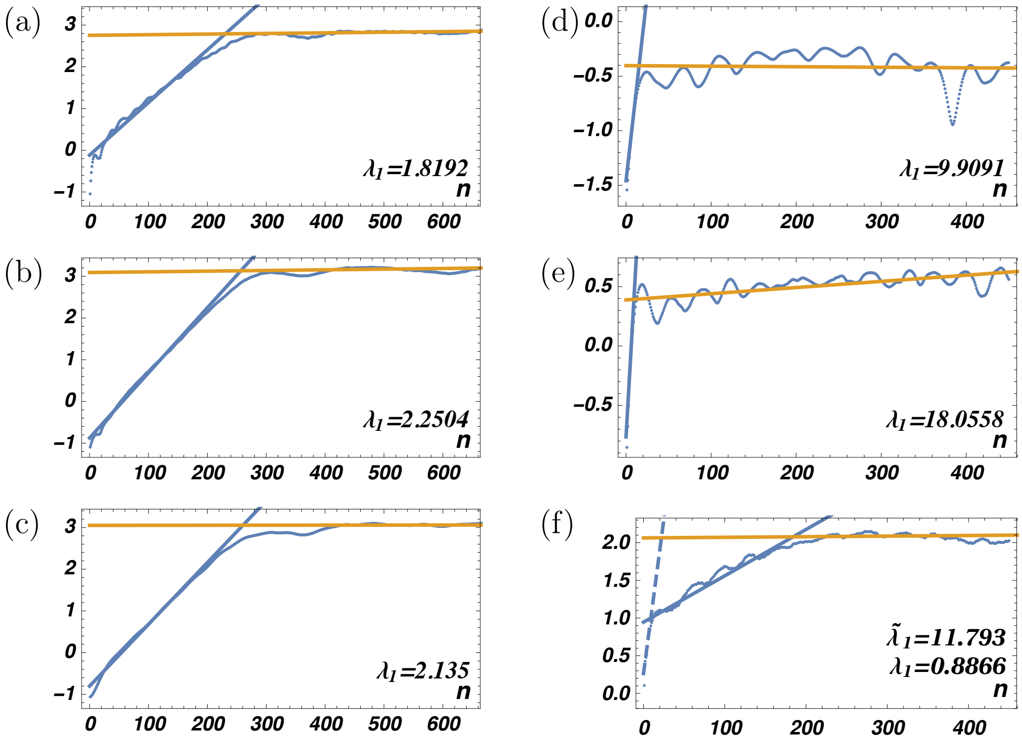

Figure 2: Separation of close trajectories in dependence on time lag $n\Delta t$ computed using the Rosenstein's algorithm.
Lorenz system: (a) 1-dimensional data, $m = 2$, (b) 1-dimensional data, $m = 3$, (c) 3-dimensional data, no dimensional reconstruction.
Experimental data: (d) 1-dimensional data, $m = 4$, (e) 2-dimensional data, $m = 4$, (f) 4-dimensional data, no dimensional reconstruction.

identified, resulting in two completely different values of the LE estimate. Moreover, it is clear in plots (d–e) that the dimensional reconstruction used in the respective cases was not sufficient and lead to the unrealistic estimate of the attractor structure. The effect of the underestimated dimension is addressed, e.g., in [3], [8] and exhibits itself by a typical shape of curves similar to cases (d-e). When compared to results obtained using data measured for other configurations of the ball, it appears that the lower value of the LE λ_1 is the more likely one.

5. Conclusions

Estimation of Lyapunov exponents is a common and important task in study of dynamical systems. In the case of discrete data, namely those obtained experimentally, the functionality of available approaches is limited and closely reflects quality of the data. It appeared that the Rosenstein's algorithm was able to identify positive character of the dominant LE in the experimental data, however, a high variability of its estimates obtained in different set ups significantly lowers credibility of the results. In the case of data originating from the Lorenz system, the Rosenstein's algorithm performed well without any interactive intervention in contrast to the case of the algorithm due to Wolf. The resulting estimate of the dominant LE was fairly accurate. As a consequence, it is necessary to make an appeal to experimenters to record data histories long enough to sufficiently fill up the phase space.

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